THE GASEOUS STUDY: AN EARLY HISTORY OF MATHEMATICAL LOGIC AND SET THEORY

A Thesis
By
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Abstract

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The historiographies of logic and set theory have told the story of crosspollination between these two strands occurring in the early twentieth century. This thesis argues for earlier cross-pollinations and traces the history of these interactions. The narrative begins with the British logicians in the first half of the nineteenth century. Then narrative then follows the history through George Boole, Bernard Bolzano, Gottlob Frege, Giuseppe Peano, Richard Dedekind, and Georg Cantor. Along the way there will be explication of the ideas that marked this history. These are put forward mainly for the non-specialist; while the appendix contains more advanced aspects of these ideas.

Acknowledgements

I would like to acknowledge the support of Dr. Michael Behrent who was the chair of my thesis committee. Dr. Phillip Johnson and Dr. Lucinda McCray, both of whom also served on my committee. I would like to acknowledge Dr. Patrick Rardin for supplying feedback on parts of the thesis. I would also like to thank Dr. Katie Mawhinney for help understanding the more difficult mathematics involved in this thesis.

Dedication

I would like to dedicate this work to my Mom and Dad, and Matt and Erica. I would also like to dedicate this work to David Browne of Brevard.

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Introduction

> "From the paradise that Cantor has created for us no one will cast us out."
> -David Hilbert[1]

> "There is no retreat in mathematics except in the gaseous part. (you may find that some of mathematics is uninteresting-that Cantor's paradise is not a paradise.)"
> -Ludwig Wittgenstein[2]

Why, when mathematicians probed the foundation of mathematics in the nineteenth century, did they develop set theory? Why set theory rose to prominence is partly a question for historians. The truth of set theory is of course determined by mathematicians, but the legitimacy of its ascension in the world of mathematical academia is determined by historians of mathematics. The historian can tell us what questions were being answered by these mathematicians, and what the philosophical influences were on the development of set theory. It is for the philosopher to tell us the merit of these influences. Set theory as discussed here will be what is called today naïve set theory. Naïve set theory concerns the generation and manipulation of sets, or collections of things. In set theory this almost always means number systems i.e., natural numbers, algebraic numbers, real numbers etc. Set theory, and so its history, is important because it acts as the foundation for many branches of mathematics today. A wide variety of sub-fields of mathematics use the idea of collections of objects. This story will be one of interaction

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[1] Akihiro Kanamori, "Hilbert and Set Theory," *Synthese* 110 (January 1997): 77.
[2] Ludwig Wittgenstein, *Wittgensteins Lectures: Cambridge, 1932-1935* (Amherst, New York: Prometheus Books, 2001), 225.

between logic and set theory. These two strains met at different points throughout the nineteenth century. This thesis will be about these meetings, and will ultimately argue that the history of such meetings is older than historians have previously thought. This story begins with the logic that came out of Britain before 1847. It continues with the British mathematician George Boole who in 1847 published the groundbreaking *The Mathematical Analysis of Logic*. This work created the new logic that would be one of the great turns in the history of logic. It would have great impact on the development of set theory. The historiography is beginning to change in a way that will be reflected in this thesis. The new view of historians is that Augustus De Morgan and Gottlob Frege should be considered cofounders of mathematical logic, along with Boole.[3] I will refer to their new logic as "mathematical." This new logic was characterized by an algebra. This means a system of symbols and their manipulations. Logic had advanced far enough that it could be operated by rules very similar to algebra. Logic could now be calculated, it was now a calculus. The turn after Boole led down a road to increased interaction with mathematics in the form of set theory. On this road logic shed the baggage of the syllogism, psychology, natural language, and exclusive disjunction. This thesis will end with Georg Cantor and the creation of the completed corpus of set theory.

Traditionally this history has largely been seen as the innovation of Georg Cantor in the field of mathematics. Historians depict set theory as an innovation that is historically isolated from logic. They tend also to see set theory as isolated from pre-Cantorian mathematics. This paper will take an approach that gives greater emphasis on the influence of and parallel development with mathematical logic. This paper will argue that George Boole and Gottlob Frege created a new logic. This logic was different from that which preceded it in its aims. Frege

[3] The historians Massimo Mugnai and Daniel Merrill have advanced these views.

and Giuseppe Peano then put a philosophical movement behind this new logic. This new philosophy found an analog in considerations of the infinite. Infinity became mixed with logic and logicism and out of this alliance came set theory. The historiography of set theory tends to begin its histories with Cantor.[4] This thesis will pick up set theory earlier with the Czech mathematician Bernard Bolzano. I argue that Bolzano framed the investigation that Cantor would make.

This paper will break from tradition in another way. It is written by an author immersed not in the field of mathematics but rather history. Traditionally investigations such as this one have been carried out by historians of mathematics who have spent their academic career in mathematics departments. The historian of science Thomas Kuhn wrote in 1968, though he spoke instead of the history of science: "Until very recently most of those who wrote the history of science were practicing scientists, sometimes eminent ones. Usually history was for them a by-product of pedagogy."[5] Kuhn's description of the condition of the history of science could be a description today for the history of mathematics. The history of mathematics when taught is often taught as a way to make mathematical ideas accessible to students. Admirable as this is, it will never lead to a sophisticated discipline. A view as to why this is important again comes from Thomas Kuhn. Kuhn, himself trained as a physicist, wrote in 1971:

> the men who study the development of a discipline from within that discipline's parent department concentrate excessively on the internal logic of the field they study, often missing both consequences and causes in the larger cultures. I remember with deep embarrassment the day on which a student found occasion to remind me that Arnold Sommerfeld's relativistic treatment of the atom was invented midway through the First World War.[6]

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[4] One need only look at the titles of books in this field, e.g., Ivor Gratten-Guiness *Search for Mathematical Roots, 1870-1940*.
[5] Thomas S. Kuhn, *The essential tension: selected studies in scientific tradition and change* (Chicago: University of Chicago Press, 1977), 105.
[6] Ibid, 153.

I am not accusing today's math historians of committing such an error. What Kuhn voices here is the danger in having experts in a field study the development of that field. So what does a historian offer to the study of nineteenth century logic? The nineteenth century was a period of strong intellectual currents and political upheaval, particularly on the continent. Analysis of this sort of extra mathematical influences is what the historian can offer.

The writer J.N. Crossley describes the early history of mathematical logic as two strains occupied by five pre-1900 thinkers. These are George Boole, Gottlob Frege, Bernard Bolzano, Richard Dedekind, and Georg Cantor[7]. In addition to these five I will provide a chapter on Giuseppe Peano. The justification for this is that Peano's *Principles of Arithmetic* was an important bridge between Boole and a consideration of the natural number system. Peano marks a combination of elements of Boole's and Frege's systems. He was, along with Frege, a founder of logicism. It was logicism that had a profound effect on the work of Cantor and Dedekind. Logic, from Frege on, had been an attempt to find a foundation for science and mathematics. Logic and set theory were after the same thing. Many parts of the logical systems I will cover were actually equivalent to what was happening in set theory. This led to the formation of a research environment between Richard Dedekind, Gottlob Frege, and Giuseppe Peano. Crossley's diagram describes the set theoretical strain crossing from Cantor to the logical strain sometime after Bertrand Russell. I will argue that this eventual cross-pollination began in the nineteenth century as a result of developments in the search for a foundation for mathematics. The logical strain began with Boole and was influenced by logicians farther back. The set

[7] John N. Crossley, *What is mathematical logic?* (New York: Dover, 1990), 2.

theoretical strain began with Bernard Bolzano's *Paradoxes of the Infinite* and was influenced by problems in the use of the infinite in mathematics.

A cursory inspection of this work will show a heavy influence on the subject of logic from Britain. Of the first two chapters C.S. Peirce is the only non-British mathematician discussed. This is indicative of dominance in the first half of the nineteenth century of British logic. In the later chapters of this work the dominance shifts to the continent, particularly Germany and Italy. This is a part of the following narrative where we must heed Kuhn's cautionary tale and look outside of the history of mathematics and into the larger intellectual trends and in some cases political trends of the day. Raymond Wilder writes:

> Mathematics does not grow because a Newton, a Riemann, or a Gauss happened to be born at a certain time; rather, great mathematicians were made because the cultural conditions–and this includes the mathematical materials–were conducive to developing them.[8]

I will show how Britain and then Germany were conducive to logic and then set theory. This thesis will look at the larger context of British and continental intellectual and political history.

While there is ample scholarship on the history of logic and set theory, the chronological focus of this work is novel. Most historical studies of set theory start with Cantor and go forward into the early 20th century. When studying mathematical logic, historians will start with Gottfried Wilhelm Leibniz and go forward to Peano, Boole, or Frege. N.I. Styazhkin's work *The History of Mathematical Logic from Leibniz to Peano* is perhaps the most cited work in this thesis. Nonetheless, Styazhkin's book suffers from just this bias. He stops short of discussing Bertrand Russell. Drawing the line at Peano, apparently Styazhkin believed that this large chunk of history had an ending in Peano. In this work there is also very little mention of Georg Cantor

[8] Raymond Wilder, *Introduction to the Foundations of Mathematics* (New York: Wiley, 1952), 278.

or Richard Dedekind. I believe this is because Styazhkin believes first, that logic from Leibniz to Peano was largely separate from set theory; and second, he believed that logic after Peano changed radically in its relationship with set theory. This is just the view that I want to combat. The reason for my approach is that mathematical logic brought up questions of where logic was supposed to fit into mathematics. This can be seen in the logicism that Frege and Peano would take up. Peano and Frege both had strong analogs in their systems to set theory. It is Peano who, among the logicians of his time, most closely shadowed early set theory. My approach as applied to set theory begins with Bernard Bolzano and continues to treatments of Richard Dedekind and Georg Cantor. Throughout this treatment we see interaction with logic. Dedekind interacted intellectually with Frege and Peano.

The chronological treatment of the subject matter adopted here is justified on two counts. First, almost all the figures were continental. This makes a national treatment unlikely since so many of these figures were German, they were almost all in close geographic proximity to each other. This sort of treatment would make for a lopsided narrative. Second, the ideas of these figures were disseminated effectively through the literature of the time. The only mathematician who did sink into obscurity for a time was Bolzano, but he was rediscovered in time for the later figures I cover to know of his work.

The concept of cross-pollination is central to this discussion. Since what is being argued for is a pre-Russellian cross-pollination between logic and set theory we should flesh this idea out. The term cross-pollination in most cases means that thinkers in these two strains read each other, but this not always the case. In some circumstances both strains actually collide, as we will see happened with Dedekind, Frege, and Peano. In this case shared results are as important as shared reading lists. Cross-pollination should be taken as generative. That is, an innovation in one strain spurs innovation in the other. This will be seen in the influence that logicism had on set theory, and the influence set theory had on the discussion of sequences in logic.

The question may justly be asked: Why is a reappraisal of the development of mathematical logic and set theory important? The first answer I will offer is for the historian; it is that we have to reconsider whether Cantor would have discovered set theory were it not for the prior researches of Bolzano and Dedekind, and the intercession of logic and logic's own development in the nineteenth century. But scholars outside of history may ask why this is important. For them I diverge into the philosophy of mathematics. I believe that the questions philosophers of mathematics deal with are largely historical in their answer. If we want to know something about how mathematics gets done we ask: How has mathematics gotten done? When the philosopher of mathematics Michael Resnik says that mathematics is pattern seeking, what we ought to do is look at a history like that presented here and ask: Was this pattern seeking?[9] If so then well and good, and we go on to the next history. If not then Resnik has a problem, and he has to either show us where we went wrong, or amend his theory. So the following treatment of history is important because this is just how questions in the philosophy of mathematics get answered. The third answer is for the mathematician. It is that the developments covered in the following pages tell us how logic can relate to mathematics in the future. Many of these figures were concerned with how the new logic could be so mathematical, and yet it not be obvious where logic fits into mathematics. Their answers can guide us in an age where logic is divided between philosophy and mathematics.

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[9] For more on Resnik's view of mathematics see: Resnik, Michael D. *Mathematics as a Science of Patterns*. (Oxford: Clarendon Press, 1997).

1: Logic Before Boole

Introduction

Logic before 1847 was of a different kind from logic after. I will begin by terming pre-1847 logic as “symbolic logic” and post-1846 logic as “mathematical logic.” Symbolic logic symbolized parts of natural language but it did not comprise a calculus. What this means is that there was no way to conduct calculations with logic. The symbolic turn in logic began with Gottfried Leibniz (1646-1716). In this chapter it means a logic that to some degree at least, allows calculations.[10] In this chapter I want to accomplish the following:

1. I want to show the state of logic before 1847 so that the accomplishments of figures discussed later will have a context for the reader.

2. I also want to argue that the logic in this chapter, with the exception of Augustus De Morgan was not “mathematical.” N.I. Styazhkin wrote of Leibniz and his founding of symbolic logic:

> Here Leibniz developed one of his favorite ideas: “an alphabet of human thought that makes it possible to deductively derive new ideas by means of definite rules for combining symbols.” Here the logical idea of pasigraphy is clearly distinguished from the linguistic idea of creating a “universal language.”[11]

Leibniz’s desire to develop “an alphabet of human thought” dominated logic until 1847, when the work of George Boole began to turn away from the domination of language and psychology

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[10] An example of a calculation in logic would be deriving the conjunctive normal form from a proposition or constructing a proof.

[11] N.I. Styazhkin, *History of Mathematical logic from Leibniz to Peano* (Cambridge, Mass: The M.I.T. Press, 1969), 65.

in logic. Logicians failed to develop mathematical logic before the year 1847. Their logic was deeply psychological and based on natural language. They were also overwhelmingly concerned with the syllogism. There were four mechanical problems with this logic; these were: quantification of the predicate, negation, disjunction, and the empty class. Each of these will be discussed with respect to the logicians Augustus De Morgan and John Stuart Mill, and several ancillary logicians.

The reason for making George Boole a turning point in the history of logic is captured well by the historian Massimo Mugnai. Mugnai ties together Leibniz and the founding of mathematical logic when he writes: "On a larger scale, however, it seems to me that Boole and Frege complete a process which began in the second half of the seventeenth century."[12] There can be little doubt that Mugnai is referring to Leibniz here. This "process" which we may take as the symbolic agenda was begun in the seventeenth century. With Boole and Gottlob Frege (1848-1925) the struggle to algebratize logic ended. This chapter then is about this struggle. Another strong characteristic of this period was the emphasis on the syllogism. One has only to glance at a logic book from the period to see page after page devoted to the syllogism. The syllogism is a three part argument in which two premises imply one conclusion. It was seen as the proper and rigorous method of thinking, the ratiocination. It was believed to be the fundamental building block of thought. The task of the logician then was to categorize the syllogisms and arrive at an inherent organization of logic. The logic of this period then was characterized by incomplete symbolization, incalculable, and centered on the syllogism.

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[12] Massimo Mugnai, "Logic and Mathematics in the Seventeenth Century," *History and Philosophy of Logic.* 31, no. 4 (2010): 311.

Logic Before 1800

This chapter will begin with a brief explication of the history of logic. The dominant force in logic for centuries had been the *Priori* and *Posteriori Analytics* of Aristotle. It is in these foundational works that Aristotle gives us the syllogism. The second great gain in logic was the work of the German mathematician and philosopher Gottfried Wilhelm Leibniz (1646-1716). Most of Leibniz's logical work was not published during his life. The search for a universal logical language that we will see in the nineteenth century was presaged by Leibniz. The first half of the nineteenth century saw expansive research in logic. This research, for the most part, took place in Britain. As will be shown in this and the next chapter there were political, economic, and intellectual currents that existed in the nineteenth century that simply did not exist in Leibniz's life time. These will be explained later but it is important now to state that Leibniz really was ahead of his time and because of this his time did not support the large multinational research environment that this thesis will chronicle. The theme we will see in this chapter is that these logicians, with the exception of De Morgan, were not mathematical because they did not fully symbolize their systems. The aim of this class of logicians was not to make logic mathematical as, we will see, Boole did. There were logical spells that these logicians were under that did not affect Boole so much. Leibniz writes in the *Monadology*:

> There are also Axioms and Postulates or, in a word, the primary principles which cannot be proved and, indeed, have no need of proof. These are identical propositions whose opposites involve express contradictions.[13]

Axioms such as Leibniz is discussing here will be used by Frege and especially Peano in the logicist program. An inchoate logicism can be said to have always existed in logic. Raymond

[13] Gottfried Leibniz, *Discourse on Metaphysics, and The Monadology* (Mineola, N.Y.: Dover Publications, 2005), 52-53.

Wilder writes: “Leibniz, for instance, showed tendencies in this direction and would probably have gone further, to judge from his stand on the basic importance of a ‘logistics’.”[14] Leibniz’s failure to write what Boole wrote must be seen as an outcome of the different aims the two logicians had. The historian of logic Massimo Mugnai argues that seventeenth century logic was not mathematical because the aims were different, not because the mathematicians were less capable. One of the aims of this thesis is to chronicle the change in the aims of logicians. Although we will be taking George Boole’s 1847 work as the start of mathematical logic it is worth repeating that there has been debate on this. The historian Daniel Merrill believes that De Morgan’s system has just as much claim to the term mathematical logic as Boole’s does.[15] In this thesis I will be agreeing with Merrill that mathematically the two systems accomplish about the same thing. I will cover Boole in depth in a separate chapter for historical reasons. His system was more influential than De Morgan’s. As we will see the founding of mathematical logic is really too complex to have any one founder.

Augustus De Morgan

De Morgan straddled the line between symbolic and mathematical logic. He and Boole were contemporaries, though De Morgan was Boole’s senior. He was a professor of mathematics at the University of London. It was this association with the newly founded university that we will see brought him into contact with exiles from other universities. The mathematician Raymond Wilder has adequately summed up the division of logic as follows:

> And, whereas logic was traditionally a cut-and-dry rehash of the work of Aristotle—with great concentration on the syllogism, etc.—it has today become a lively and

[14] Wilder, *Introduction*, 230.
[15] This view can be found in Daniel Merrill, “Augustus De Morgan’s Boolean Algebra.” *History and Philosophy of Logic*. 26, no. 2 (2005).

> growing field of investigation, known under the name of *symbolic logic* or *mathematical logic*.[16]

This was still true of De Morgan whose work was plagued by an Aristotelian like emphasis of the syllogism. Augustus De Morgan was a British logician. He was older than Boole but still a contemporary. De Morgan taught at the University of London. His most influential work, and the one that will mostly be discussed here is *Formal Logic* (1847). De Morgan had taken the opposite side of the quantification debate from Boole. Today De Morgan is being redeemed as someone who had equal, if not influential, accomplishments to Boole. Much of what Boole would accomplish in 1847 was closely aligned to investigations De Morgan was making. De Morgan is best known for his laws which state: If it is not the case that A or B is true, then we may infer that A is not true and B is not true; also if it is not the case that A and B is true, then we can infer that either A is not true or B is not true. Stanley Burris writes of De Morgan:

> Augustus De Morgan was a transitional logician, educated in the traditional logic that was solidly based on the Aristotelian syllogism, active in the reform of logic, and supportive of the new developments (of Boole) in logic.[17]

De Morgan stopped short when it came to symbolizing negation. Burris points out that De Morgan symbolized negation only once as “(F)” and afterwards simply wrote “denies F” for the negation, this prevented him from developing a complete algebra.[18] De Morgan instead handled negation almost exclusively with the complement of the class. The complement of a class is the class of things not contained in that class. The complement of the class of cats would be the class of non-cats. Thus, “not-X” would be written as “x” which is the complement of “X.”[19] Negation

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[16] Wilder, *Introduction*, 56.
[17] Stanley Burris, *Contributions of the Logicians*(2001), University of Waterloo, http://www.math.uwaterloo.ca/~snburris/htdocs/LOGIC/LOGICIANS/notes1.pdf (accessed 6/13), 27.
[18] Ibid, 28.
[19] The complement of a class is everything excluded by that class.

was couched in the complement; it could not be manipulated. There was no separate character for negation that could be manipulated. This development would have to wait for Frege.

De Morgan became embroiled in a debate over the quantification of the predicate with the Scottish logician William Hamilton. Quantification centers around the question: what do propositions about classes look like from the predicate's viewpoint? For De Morgan "all *A* are *B*" implies "all B are A." If De Morgan wanted to express "all *A* are *B,* but not all *B* are *A*" he would have to use something like Boole's "v"[20] that is he would have to write "all A are B" and "some B are A." In his *System of Logic* (1843) John Stuart Mill characterizes the debate. The debate actually pertained to who had developed a sufficient system to quantify the predicate. De Morgan was the victor, but not because he was first, but rather because Hamilton never quantified the predicate sufficiently.[21] Boole mentions this debate as the main impetus for writing *The Mathematical Analysis of Logic* (1847). Hamilton took the opposite tack and argued that "all *A* are *B*" should be interpreted as implying that "some *B* are *A*." "*A* = *B*" should be stated as "all *A* are all *B*."[22]

De Morgan's claim to co-founding math logic comes from his work *Formal Logic*. In *Formal Logic,* De Morgan confronts the problem of the null class. This problem can be stated in the following example. Take the proposition:

"All X are Y"

Does this imply that,

(1) "All Y are X"

[20] This will be discussed in the next chapter.
[21] There is some debate on this point, see the footnote in the next chapter. If interested in Hamilton's quantification theory I can suggest W. Bednarowski, "Hamilton's Quantification of the Predicate", *Proceedings of the Aristotelian Society*.
[22] John Stuart Mill, *A System of Logic, Ratiocinative and Inductive, Being a Connected View of the Principles of Evidence and the Methods of Scientific Investigation* (London: Longmans, green, and Co, 1884), 113.

Or does it imply:

(2) "Some Y are X"

Or, is it possible that:

(3) "No Y are X"

This problem is related to the debate over the quantification of the predicate. That debate was between the first two interpretations. The third interpretation was not considered possible to these logicians. There was one empty class and that class was more of an operation than an actual class. To understand this last possibility we need to consider the proposition "all unicorns are creatures." It seems that to say that the propositions "all creatures are unicorns" and "some creatures are unicorns" are absurd. But if all unicorns are creatures than how is it that no creatures are unicorns? The answer is that the null class of unicorns was being manipulated as a purely logical entity. The term "unicorn" is simply a mask donned by the empty class. Unicorns were ignored. What is important to take away from this is that logicians of the period were taking as a sort of axiom that if a class was being named it was because something fell under it. This issue would not be fully handled until the implicational logic of Frege.

Stanley Burris argues that De Morgan did not accept the empty class or universe class. This is true in so far as we mean a metaphysical universal class.[23] But, as Daniel Merrill correctly points out, De Morgan did allow for "U" to symbolize the "universe of discourse."[24] This is the logical universe of propositions. De Morgan writes in *Formal Logic*:

> If, the universe being the name U, we have a right to say 'every X is Y,' then we can only extend the universe so as to make it include all possible names, by saying 'Every X which is U is one of the Ys which are Us,' or something equivalent.[25]

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[23] Burris, *Contributions*, 4.
[24] Daniel Merrill, "Augustus De Morgan's Boolean Algebra". History and Philosophy of Logic. 26, no. 2 (2005): 81.
[25] Augustus De Morgan, *Formal Logic* (London: Taylor and Walton, 1847) https://play.google.com/books/reader?id=HscAAAAAMAAJ&printsec=frontcover&output=reader&authuser=0&hl

De Morgan means U as the universe of possible names. As with the other terms De Morgan symbolized negation with the complement of the term, he continues this practice with "U." u symbolizes the complement of the universal class. That is the empty class. The empty class is not used in categorical propositions, these are AEIO. In De Morgan's *Formal Logic* the "u" is to denote nonexistence and is only to be used when a literal translation from natural language is desired. Thus, the categorical proposition "E" can be written two ways for De Morgan but preference is given to the first[26]:

X.Y = XY)u

The right side of this equation states that the members that are common to X and Y are all members of the empty set. This is just a way of saying that the members that are common to X and Y do not exist. De Morgan's notation is to be interpreted in the following way. A period denotes disjoint classes. A comma is disjunction, this is similar to union in set theory. A parentheses, such as *P*)*Q* signifies class membership. So this would be "All *P* are *Q*." The colon represents the proposition O. So *P*:*Q* signifies "Some *P* is not *Q*."[27] The above example of "all unicorns are creatures" would then be written in De Morgan's notation as:

X)Y
X)u

Therefore   Y)x

This syllogism reads "all x's are y's," "the class of x's is empty," therefore "no y's are x's." As we saw earlier, the use of "u" has more to do with natural language for De Morgan than the

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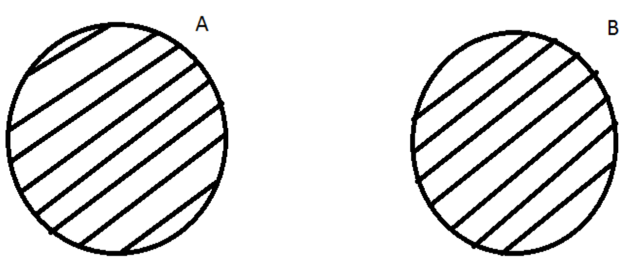
=en&pg=GBS.PR1 (accessed 11/13), 55.
[26] Ibid, 120.
[27] Ibid, 60.

actual structure of thought. It is merely a quirk of our language that we have words like "unicorn" or "chimera."

Disjunction (or) at this point was exclusive and would continue to be so throughout Boole's career. Disjunction had to be made inclusive before mathematical logic could be fully formulated. Consider the statement "either it is sunny outside, or it is not sunny outside." These possibilities exclude each other; it cannot be both sunny and not sunny. These logicians, and even Boole, took disjunction solely in this sense. Now, consider this statement "I hope it is not cold or windy today." By this we usually mean that we also hope that it is not both. Here the word "or" is doing something different than in the preceding example. It is inclusive, I also do not want it to be both cold and windy. When logicians use disjunction there are three situations which need to be symbolized; they are illustrated below along with their respective symbolizations:[28]

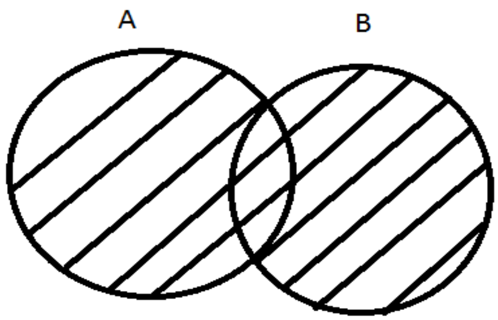


These two classes are disjoint, that is they have no members in common. In such a situation there is no difference in the use of exclusive or inclusive disjunction.

Inclusive disjunction A v B

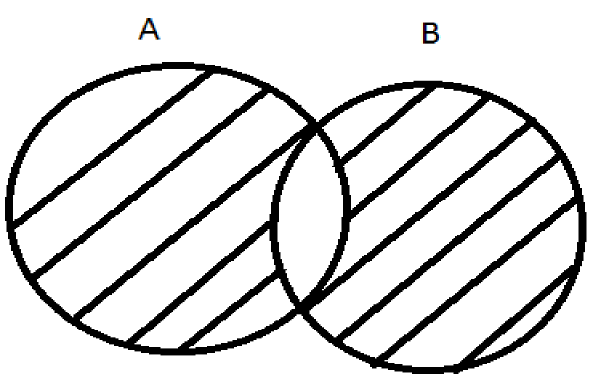
Exclusive disjunction A v B

[28] Interpret these symbols as v = disjunction, ∧ = conjunction, and ~ = negation. In the diagrams below each circle indicates a class and the striped regions contain the objects that the logician is trying to capture with his logic.



These two classes have members in common. The logician who tries to capture both classes along with the members in common will have no trouble in using inclusive disjunction. Exclusive disjunction is untenable in this situation. This is because this disjunction would have to assert both *A* and *B* and not both *A* and *B*.

Inclusive disjunction	A v B
Exclusive Disjunction	Conceptual contradiction



In this example the logician wishes to express two classes that overlap but not the members they share. Exclusive disjunction expresses this situation in the same manner that it expressed the disjoint classes above. Inclusive disjunction is more complex here and the logician is required to conjoin two propositions.

Inclusive disjunction $(A \vee B) \wedge \sim (A \wedge B)$

Exclusive disjunction A v B

Both inclusive and exclusive disjunctions run into difficulties. In the third example inclusive disjunction requires two propositions to deal with this situation, whereas exclusive disjunction requires only one proposition. However, in the second example exclusive disjunction runs into a more serious difficulty. Because exclusive disjunction states “A or B but not both” we cannot then posit a conjunction such as “A and B.” This would in effect be asserting “A or B but not both, and both.” What this misunderstanding of disjunction stems from is that these logicians were following the intuitive understanding of language too closely. They saw the use of language as containing, in situ, a logic proper. But human language makes logical mistakes. It was a realization of this fact that would spur Frege. Throughout the nineteenth century exclusive disjunction would be phased out. But even as late as 1906 H.W.B. Joseph would write of disjunction:

> In a disjunctive argument one premise is a disjunctive proposition; the other is a categorical proposition, affirming or denying one of the alternatives in the former. From these follows as conclusion a categorical proposition, denying or affirming the other alternative.[29]

It should be pointed out here that Joseph’s book contained in it a non-mathematical logic where the decision as to an exclusive or inclusive disjunction was up to the author. Frege also acknowledged both disjunctions, but he used only inclusive. Nonmathematical logic had no horse in this race.

De Morgan to a small extent foretold one-to-one correspondence by looking at the correspondence of intersections between lines drawn from the vertex to the base and a line

[29] H. W. B. Joseph, *An Introduction to Logic* (Forgotten Books, 2012), 317.

cutting the triangle parallel to the base.[30] This is a concept that will be encountered in just about every chapter of this thesis. It gains its greatest force in set theory and the proofs of Georg Cantor. It lends credence to the argument of this thesis that one-to-one correspondence is first being mentioned in a chapter solely concerning logicians. This may be taken as the first of the cross pollinations.

John Stuart Mill

De Morgan was only one of a large group of logicians working in Great Britain at this time. John Stuart Mill is perhaps known least for his work in logic. Rather, what people know of Mill comes from his two great works in philosophy, *On Liberty* (1859), and *Utilitarianism* (1861). Mill's *System of Logic* (1843) was one of the most important works of logic in this period. Mill held no academic post but rather was a clerk in the East India Company and eventual Member of Parliament. Boole cites Mill's system of logic in The Mathematical Analysis of Logic. In *System of Logic* Mill defines logic as:

> the science of the operations of the understanding which are subservient to the estimation of evidence: both the process itself of advancing from known truths to unknown, and all other intellectual operations in so far as auxiliary to this. It includes, therefore, the operation of Naming; for language is an instrument of thought, as well as a means of communicating our thoughts.[31]

What is important here is the linguistic nature of logic. Mill uses words like "understanding" and "naming," he is interested in logic which for him is thought and communication of thought. Logic then is about thought as language. What is perhaps more important is what logic is not for Mill. He writes:

> There are other more elementary processes, concerned in all thinking, such as Conception, Memory, and the like; but of these it is not necessary that Logic should

[30] Bernard Bolzano, *Paradoxes of the Infinite* (New Haven: Yale University Press, 1950), 26. See appendix for illustration.
[31] Mill, *System of Logic*, 6-7.

> take any peculiar cognizance, since they have no special connexion with the problem of Evidence, further than that, like all other problems addressed to the understanding, it presupposes them.[32]

This is a distinction that Frege would make in the *Begriffschrift*. Frege will formalize the conceptual exactly because it is more fundamental. This difference is between sentence and proposition.

Another prominent logician of the day was William Whewell. *System of Logic* was published in 1843; three years earlier Whewell published his work *Philosophy of the Inductive Sciences* (1840). In his autobiography Mill writes of the occasion of the publication of Whewell's book:

> It gave me what I greatly desired, a full treatment of the subject by an antagonist, and enabled me to present my ideas with greater clearness and emphasis as well as fuller and more varied development, in defending them against definite objections, or confronting them distinctly with an opposite theory.[33]

Mill says in his autobiography that Whewell's book was the summation of the position he was attacking. Mill's attack on Whewell's position was essentially an attack on non-empirical or non-logical propositions. This has traditionally been called metaphysics. We see here the influence the Positivism of the day had on Mill. We will see more of this later in Mill's correspondence with August Comte. Mill writes later in his autobiography:

> The notion that truths external to the mind may be known by intuition or consciousness, independently of observation and experience, is, I am persuaded, in these times, the great intellectual support of false doctrines and bad institutions.[34]

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[32] Ibid, 7.
[33] John Stuart Mill, *Autobiography of John Stuart Mill* (New York: New American Library, 1964), 163.
[34] Ibid, 164.

Mill's attack on metaphysics takes center stage in *System of Logic*. The *System of Logic* treats the concept of number. Mill asserts that a difference in quantity is empirical. This would be a point of contention by Frege. Mill writes:

> Let us imagine two things, between which there is no difference, (that is, no dissimilarity,) except in quantity alone: for instance, a gallon of water, and more than a gallon of water. A gallon of water, like any other external object, makes its presence known to us by a set of sensations which it excites. Ten gallons of water are also an external object, making its presence known to us in a similar manner; and as we do not mistake ten gallons of water for a gallon of water, it is plain that the set of sensations is more or less different in the two cases.[35]

In number Mill believes that we compare apparent or imagined sensations that make number plain to see. In a sense we imagine one and then ten gallons of water. The definitions of these numbers then are just the sensations they would cause in us. Mill goes on to write: "In like manner, a gallon of water, and a gallon of wine, are two external objects, making their presence known by two sets of sensations, which sensations are different from each other."[36] Number then is a concrete adjective, like "red," appended to a subject. Mill then writes:

> What is the real distinction between the two cases? It is not within the province of Logic to analyse it; nor to decide whether it is susceptible of analysis or not. For us the following considerations are sufficient. It is evident that the sensations I receive from the gallon of water, and those I receive from the gallon of wine, are not the same, that is, not precisely alike; neither are they altogether unlike: they are partly similar, partly dissimilar; and that in which they resemble is precisely that in which alone the gallon of water and the ten gallons do not resemble.[37]

This difference for Mill is quantity. It is not for logic to determine this. Mill would probably say that it is psychological. This is a main point of disagreement between Mill and Frege. Frege attacked Mill over this in *The Foundations of Arithmetic*.

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[35] Mill, *System of Logic*, 46.
[36] Ibid.
[37] Ibid.

Logic in the Intellectual Scene

A discussion of De Morgan gives the opportunity to reach out of the history of logic for a moment and discuss the larger intellectual history taking place in Britain. There was a general intellectual current that the logic discussed so far was a part of. The historian Joan Richards writes of De Morgan's program in mathematics:

> The roots of this program were embedded in the eighteenth century, when a determined group of Englishmen struggled to reform the Anglican Church. Over the course of the 1770s and 1780s, Unitarians like Theophilus Lindsey, Joseph Preistly, and, in the younger generation, William Frend worked to define the parameters of a rational religion.[38]

William Frend was a Unitarian preacher who was also a staunch pacifist. He had been kicked out of Cambridge University and met De Morgan at the University of London. There De Morgan was a professor of Mathematics. He befriended Frend and even married his daughter Sophia.[39] What could bring a mathematician together with a Unitarian preacher and political apostate? There was a common belief in British intellectual circles that rational thought could mitigate the ills of society. Logicians found a role in solving society's ills and this is why in this period there was such an emphasis on ratiocination. De Morgan wrote:

> This language, without reference to any of its applications, is instrumental in furnishing the mind with new ideas, and calling into exercise some of the powers that most peculiarly distinguish man from the brute creation.[40]

For De Morgan reason is what made humans better able to survive in the world. His logic then was a way to enhance this human faculty for the betterment of British society. John Stuart Mill also saw logic as a panacea to society's problems. Mill titled chapter six of *System of Logic* "On

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[38] Joan Richards, "'This Compendious Language': Mathematics in the World of Augustus De Morgan" *Isis* 102 (September 2011), 508.
[39] This account was taken from Joan Richards' Isis article and the book: Benjamin Wooley, *The Bride of Science: Romance, Reason, and Byron's Daughter* (New York: McGraw-Hill, 1999).
[40] Richards, "Compendious Language", 509.

the Logic of the Moral Sciences" and includes chapters of fallacies of practical reasoning. Mill is himself an example of the reach of logic outside of itself. Mill carried on a long correspondence with the French Positivist thinker August Comte. And from these letters it can be inferred that Mill was also Positivist. Positivism was also imbued with this aim of social betterment. The historian Geoffrey Bruun writes of Positivism: "The prestige of the priests and the philosophers dimmed before that of the scientists."[41] The question for us is: What role did logic have in positivism? This can be answered by looking at letters from around the time Mill published *System of logic* and sent a copy of it to Comte. In his May 16th 1843 letter Comte responds to *System of Logic*.[42] In this letter Comte states that he agrees almost in full with Mill's system except that he questions the probability logic Mill introduces. Mill states a quality of probability that Comte may have found unacceptable. In *System of Logic* Mill, Quoting the French scientist Pierre-Simone Laplace, writes: "'Probability,' says Laplace, 'has reference partly to our ignorance, partly to our knowledge…'."[43] Mill later writes: "We must remember that the probability of an event is not a quality of the event itself, but a mere name for the degree of ground which we, or some one else, have for expecting it."[44] Probability, Mill admits, is the mathematics of ignorance. One can see here why Comte would have a problem with probability. As Mill freely admits, probability does not touch the world. It does not investigate the world but only replaces the investigation with calculated guessing. This goes against positivism. The logic of probability was pursued more vehemently in Britain than in Comte's France. One recalls the British logicians John Venn and George Boole, who will be discussed in the next chapter.

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[41] Geoffrey Bruun, *Nineteenth Century European Civilization 1815-1914* (New York: Oxford University Press, 1960), 12.
[42] This correspondence can be found in John Stuart Mill and August Comte, *The Correspondence of John Stuart Mill and August Comte*, ed. Oscar Haac (London: Transaction Publishers, 1995), 149-158.
[43] Mill, *System of Logic*, 350.
[44] Ibid, 351.

Nonetheless, the point here is that Comte and Mill had more reason to come together in friendship over the betterment of society than to fight over the legitimacy of probability. One could say the same for De Morgan and William Frend. But this view of the purpose of logic may have had consequences. This pragmatic approach is perhaps why logic did not hit full mathematical stride in Britain but rather in Germany and Italy.

Conclusion

Logic in this period posed the questions that Boole would answer in 1847. The answer would revolutionize logic. They would also overturn much of the program outlined in this chapter. These were disjunction, the null class, the concept of number, emphasis on natural language, emphasis on psychology, and negation. These thinkers were occasionally ahead of their time. De Morgan posited univocal correspondence which would become a central theme in the theory of sets throughout the nineteenth century.

2: The Reflected Image of Logic

Introduction

Heidi White and Michael Shenefelt outline the difference between the symbolic logic discussed so far and the new mathematical logic when they write: “Because ordinary words are excluded, our knowledge of what any words mean is excluded too. This is what makes symbolic logic fundamentally different from the logic of the past.”[45] A large part of the logic discussed so far, with the exception of De Morgan’s system, was not fully symbolized. These first two chapters are about the period when logic became fully symbolized and hence mathematical. The story of this symbolization, in part, has already been told in the preceding chapter. The historian Daniel Merrill argues that De Morgan co-discovered the algebra of classes. Boole gets his own chapter because he was more widely read by later logicians than De Morgan was. Of the systems George Boole and Augustus De Morgan developed, De Morgan’s system sank into anonymity, while it was Boole’s system that was engaged with by mathematicians in Europe and America. George Boole (1815-1864) was born in Lincolnshire, England. He received no university education and instead taught at private schools. He started his own school where he was headmaster. Boole’s mathematical ability would catch up with him though, and he was lured to academia. Academia here took the form of the University of Cork in Ireland. Boole died a relatively young man in 1864.

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[45] Michael Shenefelt and Heidi White, *If A then B* (New York: Columbia University Press, 2013), 213.

As novel as his logic may have been, there were many biases that remained in Boole's system. His disjunction was exclusive and he had no means of negation. His main accomplishments lay in depsychologizing logic and in conceiving an interpretation of logical structures in a way that allowed logic to be treated like a branch of mathematics. For Boole logic was about the combination of symbols, not what the symbols meant. Boole was influenced by the logicians discussed in the last chapter. Early in *The Mathematical Analysis of Logic* he credits the quantifier debate between De Morgan and Hamilton with being the impetus for *The Mathematical Analysis of Logic* (1847). Boole was divorced from these logicians, he did not straddle the logics as De Morgan did. He was not as beholden to the syllogism as De Morgan was. It is true that De Morgan was less concerned with the syllogism than his predecessors, but he was not as able to break from it as Boole was. What Boole came to realize was that the syllogism indicated no underlying structure of logic, it was just something one could do with logic.

Boole's System

It was complete symbolization and algebraization that was the mark of the new mathematical logic that Boole brought to the world. Boole's logic was published in three works *The Mathematical Analysis of Logic* (1847), a short paper titled "The Calculus of Logic" (1848), and *The Laws of Thought* (1854). *The Mathematical Analysis of Logic* was the first exposition of what would come to be called Boole's system, but by the 1850's Boole had become dissatisfied with his 1847 and 1848 treatment and set out to put forth his entire system; this effort would culminate in *The Laws of Thought.* Boole states in the *Laws of Thought* that "the operations of Language, as an instrument of reasoning, may be conducted by a system of signs composed of the following elements"[46] Boole goes on to list these elements as "Literal signs," "Signs of operation," and "The sign of identity." The literal signs consist of variables, the signs of operation consist of the logical operators, and the sign of equation is just "=." Boole's system was an algebra of classes. What is also central to Boole's work is the interpretation of traditional symbolic logic into classes. Taking the variables *a* and *b* for classes *ab* is logical multiplication. This can be translated to English as describing "the class of objects that are common to both *a* and *b*." For disjunction Boole used the addition sign.[47] *a* + *b* then would be translated into "the class of objects that are common to *a* or to *b*." Boole's disjunction was exclusive. In every case of disjunction Boole imagines disjoint classes. This is because he wanted logical addition to be closely analogous to arithmetical addition. Logical disjunction of disjoint classes is analogous to arithmetical addition. If classes are allowed to overlap at all the analogy breaks down. Boole handles the complement of a class by positing the symbol 1 to stand for the universal class. The complement of a class would be handled by subtracting the class from the universal class as in (1-*a*) which means the class of not-*a'*s.[48] 0 is used as the empty class. Therefore, Boole would symbolize the proposition A as *ab*=*a*. He would symbolize proposition I as *ab*=v, "v" being an overlap. Boole writes:

> If some Xs are Ys, there are some terms common to the classes X and Y. Let those terms constitute a separate class v, to which there shall correspond a separate elective symbol v, then
>
> $$v = xy,$$
>
> And as v includes all terms common to the classes X and Y, we can indifferently interpret it, as Some Xs, or Some Ys.[49]

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[46] George Boole, *The Laws of Thought* (New York: Dover, 1951), 27.
[47] This will often be referred to as logical addition.
[48] Although Boole never does this you can look at this as the conjunction between the universal class and the class of not-a's. This would be closer to the modern idea of negation.
[49] George Boole, *The Mathematical Analysis of Logic* (Bristol, England: Thoemmes Press, 1998), 21.

This new “v” along with his class interpretation of the traditional copulas meant that Boole could express “some A are B,” and proposition O as “some A are not B” algebraically respectively as “AB=v” and “A(1-B)=v.” As the historian William Kneale points out, Boole’s notation led to confusion. When one writes “*ab*=v” and “*cd*=v” one is tempted to say “*ab*=*cd*.”[50] Boole would symbolize proposition E, such as “no *a* are *b*,” as *ab*=0. Boole could write “*ab*=*a*” and “*a*” could be a subset or identical with “*b*.” De Morgan demanded that “*a*” be identical with “*b*.” If Boole wanted to express identity as De Morgan did, he could write “*ab*=*a* and *ba*=*b*” as he does in *The Mathematical Analysis of Logic*. This is how Boole handles the quantification of the predicate. In “v” we can see the grasping for the existential quantifier. This would be posited by the Italian mathematician Giuseppe Peano.

Boole developed rules for classes and in the case of the index law Boole has to break from his algebraic program in logic. Burris makes the point that Boole’s index law “$x^2=x$” requires that “*x*” be a class and nothing else. Boole says himself that this is the only deviation from algebra. Boole was concerned with natural language; he wanted to represent what language is about in a rigorous way. Logic should show us, if anything, how language should be, not how it is. This is the ratiocinative program. Boole writes in *The Mathematical Analysis of Logic*:

> The theory of Logic is thus intimately connected with that of Language. A successful attempt to express logical propositions by symbols, the laws of whose combinations should be founded upon the laws of the mental processes which they represent, would, so far, be a step toward a philosophical language.[51]

The emphasis on language would decline as mathematical logic became concerned with questions fundamental to logic. Psychologism and language are connected. Boole writes:

> For though in investigating the laws of signs, *a posteriori*, the immediate subject of examination is Language, with the rules which govern its use; while in making the internal processes of thought the direct object of inquiry, we appeal in a more immediate

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[50] William Kneale,“Boole and the Revival of Logic” *Mind*, Vol. 57, No. 226 (Apr., 1948), 165.
[51] Boole, *Mathematical Analysis*, 5.

> way to our personal consciousness.-it will be found that in both cases the results obtained are formally equivalent.[52]

Boole was the forerunner, not the antagonist, to the more vocal anti-psychologism of Frege. N.I. Styazhkin writes of Boole's philosophy of mathematics when he says:

> Their analysis usually leads to the conclusion that Boole can be considered a forerunner to formalism of the type associated with Hilbert with elements of psychologism in the spirit of John Stuart Mill.[53]

Boole himself writes:

> They who are acquainted with the present state of the theory of Symbolical Algebra, are aware, that the validity of the processes of analysis does not depend upon the interpretation of the symbols which are employed, but solely upon the laws of their combination.[54]

This quote is indicative of Boole's supposedly formalist views. NI Styazhkin dismisses Boole's formalism. He argues that Boole did indeed intend logic to be fundamental to mathematics. Styazkhin argues for a proto-logicism in Boole. He dismisses Boole's psychologism when he writes: "Boole was a complete stranger to subjectivism; he recognized the existence of laws of thought as objective compulsory relations, independent of the will of the apprehending individual."[55] Styazhkin is a little off the mark. Boole believed what he studied was part of psychology, but an immutable part. It did not change from person to person. His system might better be called mathematical psychology. Boole's system was not logicism, but he does not denounce such a program. What is most accurate to say on this count, is the historian's answer; that Boole and logicism were not in the same historical theater, Boole having died in 1864.

An essential program of mathematical logic in the nineteenth century was an answer to the question: how does logic interact with mathematics? Logicians felt that if logic had come to

[52] Boole, *Laws*, 24-25.
[53] Styazhkin, *History of Mathematical Logic*, 174.
[54] Boole, *Mathematical Analysis*, 3.
[55] Styazhkin, *History of Mathematical Logic*, 175.

resemble mathematics in just about all is rules and operations then it was fair to ask what the subject of logic was. As we will see mathematical logic took several different views of this but what was constant is an emphasis on this question. Massimo Mugnai describes Boole's position on this question:

> Boole considered logic to be a branch of mathematics. For him the notion of class plays a central role in logic, and he conceives of logic as an activity centered on combining classes according to some well-specified operations.[56]

The view that logic is just another branch of mathematics brings to mind what was said by the Austrian philosopher Ludwig Wittgenstein when discussing Bertrand Russell's work in logic "Russell's calculus is not fundamental; it is just another calculus."[57] This is one side of a fundamental debate in nineteenth century logic; the other side would posit that logic was not part of but rather fundamental to mathematics. Boole did not consider his logic fundamental because to the contrary it was a mimicry of mathematics. Boole created an algebra of classes, that is, what its subject was. However, mathematical logic was just another algebra. Boole believed that with his work mathematics was no longer the science of quantity. So for him mathematical logic also had an effect on mathematics. Boole still believed that logic was part of mathematics. Since mathematical logic was certainly no science of quantity this collapsed the centuries old quantity paradigm in mathematics. Mathematics could no longer be thought of as only a science of quantity. We will see in the following chapters how mathematical logic came to be more than just a calculus for the mathematicians working on it.

As seen in the above quote from Boole, logic was about the combination of symbols, not what the symbols stand for. This idea was central to the creation of mathematical logic and was the beginning of what would develop into a whole new way of looking at logic. This is a nascent

[56] Mugnai, "Logic and Mathematics", 311.
[57] Wittgenstein, *Lectures*, 205.

viewpoint that would be the seed of Frege's rejection of the psychological in logic. One recalls Mugnai's statement at the beginning of the previous chapter, this was one of the processes that began with Boole and ended with Frege. The historian William Kneale states that Boole depsychologized logic.[58] The depsychologizing of logic is a mantle that would be taken up even more adamantly by Gottlob Frege. Depsychologizing logic meant that logic was no longer trying to mimick thought, or even to correct it, it was now more important that logic be a consistent system of symbols , operations, and their combination.

Boole handles the quantification of the predicate with the forms *ab*=*a* and *a*=*b*. Thus, Boole differentiates Hamilton's "All A are some B" and "All A are All B." Speaking of William Hamilton N.I. Styazkin writes: "His theory of quantification, complete, though not original, has to be considered among the stimulating prerequisites that opened the way for George Boole's logical calculus."[59][60] Boole writes in *The Mathematical Analysis of Logic*:

> In the spring of the present year my attention was directed to the question then moved between Sir W. Hamilton and Professor De Morgan; and I was induced by the interest which it inspired, to resume the almost-forgotten thread of former inquiries.[61]

Boole disagrees with De Morgan's quantification of the predicate when he writes:[62]

All Xs are Ys
All Ys are Xs $\quad x = y$

All Xs are Ys $\quad x(1-y) = 0$

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[58] Kneale,"Revival of Logic", 174.
[59] Styazhkin, *History of Mathematical Logic*, 156.
[60] It may be recalled that in the last chapter it was mentioned that Hamilton's QP was not complete. This is the view of William and Martha Kneale in their work *The Development of Logic*. Their argument comes from Hamilton's introduction of the term "any." Styazhkin asserts that Hamilton's quantification was complete. This debate centers around Hamilton's interpretation of "any." Both Styazhkin and Kneale agree that Hamilton takes this as equivalent to "all." Take the proposition "all *a* is not all *b*" versus "any *a* is not any *b* ", Hamilton equates these two and this is where Kneale(353) says that Hamilton went wrong. The second position seems to be describing the state of disjoint classes. The first seems to be asserting only that the situation $a = b$ is not the case.
[61] Boole, *Mathematical Analysis*, 1.
[62] Ibid, 25.

Here we see that Boole interprets "all X's are Y's " as meaning that only some Y's are X's. To say that X and Y are equivalent classes is interpreted as two statements. The problem of the quantification of the predicate is this, that, for example, when we use Boole's notation "all A are B" is symbolized as AB=A. This can further be translated as "that which is common between A and B is just A." Let's take an example:

AB=A

can be translated with three examples:

all zombies are human

all men are human

all humans are human

The problem arises when we switch the A and B and ask what is common to B and A. We get three following equations:

BA=0 the class of things common to humans and zombies is the empty class

BA=v the class of things common to humans and men is some humans

BA=B the class of things common to humans and humans is the class of humans

Boole states that BA=v is the common interpretation of AB=A. He handles the "class of all things common to humans and humans is the class of humans" with AB=A and BA=B. When it comes to zombies Boole waives his hands and simply disallows A or B to ever represent an empty class. Someone may object here that we could simply interpret "all zombies are human" with AB=0. This would put a rather severe restriction on our system though, since we could not discuss concepts. Conceptually— that is by definition—all zombies are human. What we want to say about zombies and humans and what we want to say about giraffes and humans is different. One case is disjoint, the other inexpressible.

Boole cannot accept the empty class in a proposition like *ab*=*a*, if he is to accept that "some *a* are *b*" is written as *ab*=v in his system. If he did accept the possibility of the empty class in this case, then the top proposition would have to be some sort of conceptual proposition whereas the bottom proposition would be talking about a populated class. Does Boole believe that a proposition that involves an empty class simply cannot be written or does he believe that the proposition "all unicorns are animals" should be written as ab=0, that is it should be treated in the same way as disjoint classes? If the second option, then how would he handle a situation in which both A and B are empty, such as "all existent unicorns are existent mythical animals?" As with disjunction, Boole makes a judicious decision here. This time he will only deal with populated classes. The empty class that Boole allows is one in number and it is 0; variables cannot be empty classes. He must accept that *a*, *b*, v are classes that have members.

The empty class is an important concept as is seen with the null set in set theory. Stanley Burris makes this point in the expression "XY=X therefore some Y is X (or YX=v)." This cannot be accepted unless one disallows empty classes or states that the expression is false when "X" is an empty class. Burris describes Boole's handling of the empty class in this case: "In 1847 Boole translates "All X is Y" as the equation xy = x, and when X is empty this becomes the equation "0y = 0," which is true in his system."[63] Boole was able to handle empty sets in a better way, in terms of having a logic be mathematical.

> By a class is usually meant a collection of individuals, to each of which a particular name or description may be applied; but in this work the meaning of the term will be extended so as to include the case in which but a single individual exists, answering to the required name or description, as well as the cases denoted by the terms 'nothing' and 'universe' which as 'classes' should be understood to comprise respectively 'no beings,' 'all beings,'[64]

[63] Burris, *Contributions*, 4.
[64] Akihiro Kanamori, "The Empty Set, the Singleton, and the Ordered Pair" *The Bulletin of Symbolic Logic*, Vol. 9, No. 3 (Sep., 2003), 274.

For Boole the term “class” included the concept of the empty class. Boole treats merely populated classes differently from 1 and 0. This is his solution to the problem of unicorns. As Stanley Burris stated, the class of unicorns is just 0.

Disjunction Becomes Inclusive

It was mentioned in the first chapter that the generation of mathematicians before *The Mathematical Analysis of Logic* was published used exclusive disjunction. Exclusive disjunction states the possibility of one thing or another but not both; for example, “either it will rain tomorrow or it won’t.” Boole retained exclusive disjunction keeping in line with logicians before him. Boole’s disjunction was also linguistic, meaning that it mirrored the way he thought disjunction was used in natural language. When disjunction is used in everyday speech it is as exclusive disjunction. That is, for Boole disjunction applied only to disjoint classes. For Boole it was a tool for combining two classes into one. When he discusses disjunction in *Laws of Thought* Boole is always talking about disjoint classes. Styazkhin writes:

> Addition, denoted by the symbol +. In the calculus of classes, Boole’s formula x+y corresponds to the union of the classes x and y minus their common portion; in the propositional calculus it is so-called “strict disjunction.”[65]

The preceding chapter left the question of disjunction hanging. Boole’s calculus used exclusive disjunction. Inclusive disjunction would not come until the 1860’s. The first work to do this was the 1864 book *Pure Logic* by the British logician William Stanley Jevons, a former student of Boole’s. This work was published the year Boole died. The second work was the 1867 paper “On an Improvement in Booles Calculus of Logic” by the American Charles Sanders Peirce. Jevons writes of disjunction:

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[65] Styazhkin, *History of Mathematical Logic*, 177.

> Take for instance, the proposition-'A peer is either a duke, or a marquis, or an earl, or a viscount, or a baron.' If expressed in Professor Boole's symbols, it would be implied that a peer cannot be at once a duke and a marquis, or marquis and earl. Yet many peers do possess two or more titles.[66]

Here Jevons makes the criticism of Boole's system that it does not reflect how language is used. Jevon's disjunction was also linguistic. What he disagreed with Boole about was language itself. Here we see the still strong concern with language and the psychologism that Frege would attack. Jevons chooses another example, this time from literature: "Milton has the expression in one of his sonnets-'Unstain'd by gold or fee,' where it is obvious that if the fee is not always gold, the gold is a fee or bribe."[67] Sonnets and the makeup of parliament aside, Jevons is fair in his criticism. Jevons offers another criticism of Boole and it is worth covering since it also deals with disjunction. Jevons writes:

> The term x, in his system, means all things with the quality x, denoting the things in extent, while connoting the quality in intent. If by 1 we denote all things of every quality, and then subtract, as in numbers, all those things which have the quality x, the remainder must consist of all things of the quality not-x. Thus $x + (1 – x)$ means in his system all x's with all not-x's, which, taken together, must make up all things, or 1. But let us now attempt by multiplication with x, to select all x's from this expression for all things.
>
> $$x(x + 1 – x) = x + x – x$$
>
> Professor Boole would here cross out one + x against one – x, leaving one + x, the required expression for all x's. It is surely self-evident, however, that $x + x$ is equivalent to x alone, whether we regard it in extent of meaning, as all the x's added to all the x's, which is simply all the x's, or in intent of meaning, as either x or x, which is surely x. Thus, $x + x – x$ is really 0, and not x, the required result, and it is apparent that the process of subtraction in logic is inconsistent with the self-evident Law of Unity.[68]

There is a contradiction here. Jevons is pointing out that Boole treats $x + x$ as a numerical accumulation for which when x is then subtracted remains x. But this violates Boole's own law as when he writes in *The Mathematical Analysis of Logic*, $xx=x$.[69] Boole, following logic too

[66] William Stanley Jevons, *Pure Logic and Other Minor Works* (New York: Burt Franklin, 1971), 68.
[67] Ibid, 69.
[68] Ibid, 74.
[69] Boole, *Mathematical Analysis*, 17.

closely, violates what Jevons calls the age old law of unity, x + x = x, and he violates his own law, what he calls the index law.[70] He states this law as:

> If from a group of objects we select the X's, we obtain a class of which all the members are X's. If we repeat the operation on this class no further change will ensue: in selecting the X's we take the whole.[71]

Boole believed this law to be the only difference between logic and algebra:

> The essential difference between logic and algebra, according to Boole, consists in the fact that in the former the law xx=x (the principle of idempotence) always holds, whereas in the latter the equation x^2=x holds only if x=1 or x=0, the two roots of the equation x^2-x=0.[72]

Jevons would suggest that Boole would contradict himself by agreeing with the equation xx-x=x. Jevons would argue that truer to the spirit of logic would be to say xx-x=0. Styazkhin writes:

> Jevons felt that the Boolean definition of the operation of addition, x + y, in proposing the exclusion of the common portion of the classes x and y, entailed insurmountable difficulties since it was unclear, in the case, how to interpret expressions of the type x + x.[73]

What Jevons is really getting at though is that what we are witnessing is the friction between algebra and a logic of classes. As Jevon's writes towards the end of *Pure Logic*: "Boole's system is like the shadow, the ghost, the reflected image of logic, seen among the derivatives of logic."[74]

Peirce's criticism from 1867 is more technical. Peirce states inclusive disjunction in his paper when he writes in a foot note: "i.e., a+,b is the class of those things which are a not-b, b not-a or both a and b."[75] Peirce goes on to state the this in a more formulaic way: "a+b=(a+,b)+(a,b)."[76][77] Peirce's concern here is that when adding classes, if there is overlap, the

[70] It is important to point out that Boole never stated idempotence as x + x = x, but always as xx=x or x^n=x.
[71] Boole, *Mathematical Analysis of Logic*, 17.
[72] Styazhkin, *History of Mathematical Logic*, 179.
[73] Ibid, 189.
[74] Jevons, *Pure Logic*, 77.
[75] In Peirce's paper the , occurs beneath the arithmetic operator to denote the logical operator.
[76] Charles Sanders Peirce, "On an Improvement in Boole's Calculus of Logic" *Collected Papers of Charles Sanders Peirce, vol 3-4* (Cambridge, Mass: Belknap Press, 1933), 9.
[77] , denotes conjunction in Peirce's system.

only way to get an analog to the arithmetic operation of say 2+2=4 is with inclusive disjunction. Styazhkin states the arithmetic operator "+" as follows, "a + b denotes the class of those things which are either a or b but not both simultaneously."[78] Styazhkin goes on to write: "We can see that Peirce considered ordinary disjunction to be logical, and strict Boolean disjunction to be an arithmetic operation."[79] Pierce's criticism of Boole's system does not end at disjunction. He sketches three areas where Boole's system is inadequate. Of these three the first is particularly important for our discussion. Peirce points out the inability of Boole's system to speak about one member of the class, he writes:

> First. Boole does not make use of the operations here termed logical addition and subtraction. The advantages obtained by the introduction of them are three, viz., they give unity to the system; they greatly abbreviate the labor of working with it; and they enable us to express *particular* propositions.[80]

Peirce recognized that when Boole uses "v" it is not in a way that allows us to say "some *a*." Rather, it is only used to point out that, for example, two classes partially overlap each other. *ab*=v is to communicate just that neither *a* nor *b* contain each other, but that they have some part in common. The first advantage Peirce points out was mentioned by Jevons. Peirce is pointing out the lack of cohesiveness of the half logical half algebraic system Boole has constructed. The second advantage has also already been mentioned and that is that disjunction involving overlapping classes is now analogous to arithmetic. The final advantage is that Peirce's improvements allow for quantification.

Boole and British Logic in the Larger Historical Context

The years and location of these first two chapters hint at a larger fact in this history. These two chapters, with the exception of C.S. Pierce, involve British logicians from the years 1800 to

[78] Styazhkin, *History of Mathematical Logic*, 178.
[79] Ibid.
[80] Peirce, "Boole's Calculus of Logic", 13.

1850. The historian Volker Peckhaus states that Britain saw a rebirth in the early 19th century beginning with the work of Richard Whately. Peckhaus writes:

> One can even say that neglect of formal logic could be regarded as a characteristic feature of British philosophy up to 1826 when Richard Whately (1787-1863) published his *Elements of Logic*.[81]

The years from 1826 to 1847 saw a steady increase of research in logic, much of which was outlined in the previous chapter. In that chapter we saw that the logicians up to Boole were engaged in a program of bettering society and the 1850's and 60's had this in common with the first half of the nineteenth century. The historian Geoffrey Bruun writes of the decades after 1847 that: "A strong conviction had grown up that everything in the physical universe was behaving in a rational manner and that it was man's disorderly mind that had led him to misread her."[82] There was also still a strong current of positivism in Europe that was perhaps even more at home than in the years leading up to the publication of *The Mathematical Analysis of Logic*. The artist Gustave Courbet summed up this positivism when he said: "Show me an angel and I will paint one."[83]

There is also a political context that logic took place in. Boole represents the highest point British logic would achieve in the nineteenth century. This is perhaps because of the relative political unrest on the continent in the first half of the nineteenth century. This unrest would come to a head in 1848 when the continent would erupt in nationalist movements. Although these movements were unsuccessful they began a narrative that would end in nationhood for Germany and Italy in the 1860's and 1870's. Bruun writes of Britain: "To Great Britain the middle years of the nineteenth century brought domestic tranquility, increasing

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[81] Volker Peckhaus, "19th Century Logic between Philosophy and Mathematics" *The Bulletin of Symbolic Logic* 5 (Dec., 1999), 436.
[82] Bruun, *Nineteenth Century*, 142.
[83] Ibid, 148.

prosperity, and a worldwide prestige."[84] In opposition to the stability in Britain, Bruun writes of the relative unrest on the continent: "All the fervor, the fighting, the compromising and constitution-making of 1848-9 ended in central Europe with a virtual restoration of authoritarian principles."[85] The continent erupted in revolution in 1848. But, these nationalist movements failed. British political tranquility is a possible explanation why the island was so dominant in logic from the years 1826 to 1870. Revolutions distract from logical research. They distract because it is difficult to do logical research in times of civil unrest. They redirect students and professors from logical research, in favor of radical politics.

However, there is another factor. Michael Shenefelt and Heidi White suggest that it was the industrial revolution that was the impetus behind Britain's revival and dominance of logic.[86] They argue that the Industrial Revolution convinced logicians of the power of mechanized thought. They write: "The Industrial Revolution convinced large numbers of logicians of the immense power of mechanical operations."[87] It was this machinery that filled the heads of logicians with dreams of similarly mechanized thought. The most faithful expression of this was the analytical engine of Charles Babbage. This impetus for bettering society, mentioned in the first chapter, melded with an analogue from the factory floors of the Industrial Revolution. After this chapter we will see a switch to continental Europe where logical research took a different form. So what did cause the new logic to arise? It was mentioned in the introduction that Boole himself states in *The Mathematical Analysis of Logic* that the quantifier debate between Hamilton and De Morgan was the impetus for writing that work. It is worth pointing out that De Morgan was intimately connected to this debate and also assisted the founding of mathematical

[84] Ibid, 107.
[85] Ibid, 83.
[86] This argument is located in Shenefelt and White, *A then B*, 205-234.
[87] Shenefelt and White, *A then B*, 205.

logic. It is not surprising then that Boole would have thought the debate important enough to mention in his 1847 work. But it doesn't seem that this was a cause of the new logic. It is true that Boole was a mathematician who tackled logic. But that is also true of Johan Lambert. The cause for the rise of the new logic was three part. First, as mentioned above there was the new analog of machine floor. Second, there was Boole's approach which because of his background was much more mathematical than his contemporaries. Third, in the case of both De Morgan and Boole there was incitation of the Hamilton-De Morgan debate. While I do not believe that the quantification of the predicate pointed in the direction of mathematical logic, I do think it spurred both men to develop sweeping systems of logic.

Conclusion

Boole is the founder of mathematical logic on two counts. First, that he along with De Morgan had fully algebratized logics. Secondly, of these two, Boole's system was engaged with much more than De Morgan's. Boole can perhaps be given credit for what his system lacked. It was the shortcomings of Boole's calculus that was the catalyst for the logicians that are discussed in the following chapters. Boole, nonetheless, frames the accomplishments of later logicians. This is because it is Boole with whom later logicians will have to engage. It is Boole in the logic strain and, as we will see in the next chapter, Bolzano in the set theoretical strain; who will frame the investigations in both these fields for the rest of the nineteenth century.

3: Bolzano and the Paradoxes of the Infinite

Introduction

The Czech mathematician Bernard Bolzano is where this investigation will pick up the thread of concerns that led to set theory at what can best be considered its beginning. Bernard Bolzano will be taken as the beginning of the set theoretical strain which includes Dedekind and Cantor. These pages will justify taking Bolzano as the beginning of the set theory strain. Bolzano has been neglected in the historiography. I will show in later chapters that he was he was important to both Cantor and Dedekind. These pages will also argue that there was an early link between logic and set theory in the work of Bernard Bolzano (1781-1848). Specifically, this chapter will look at the paradoxes and outline central issues that were being dealt with. This chapter will also argue that these paradoxes and Bolzano's solutions of them sets the discourse for the notion of infinity for the rest of the nineteenth century. Even the solutions that sank into obscurity will serve to show the single mindedness of *Paradoxes* that made it revolutionary.

Bolzano was the first to treat infinity in the way that later set theorists treated it, as an object of study in its own right. The best place to begin this investigation is with the book *Paradoxes of the Infinite,* published four years after *The Mathematical Analysis of Logic*. Bolzano died in 1848 and it is probable that he had not read Boole's work. *Paradoxes of the Infinite* was published posthumously through the effort of Bolzano's friend Dr. Prihonsky in 1851. The historian Ivor Gratten-Guiness argues that Bolzano made discoveries in analysis in the period Gratten-Guiness calls “new analysis.” This was a period of rapid development in the field of analysis, this research mostly took place in Paris in the early nineteenth century. Although Bolzano was in Bohemia and research in the new analysis was mostly taking place in Paris, Gratten-Guiness argues that Bolzano’s ideas arrived onto the Paris scene through the mathematician Augustin-Louis Cauchy.[88] In *Paradoxes of the Infinite* Bolzano cites the work of Cauchy and Joseph-Louis Lagrange, two of the giants in the history of analysis. What is important to draw from this is that this strain largely emerged from analysis. Bolzano was steeped in the world of analysis, and he personally made important discoveries in this field. Bolzano foresaw many of the concepts that set theory later developed. He foresaw problems that set theory tried to solve some forty years later, one being: how to tell if two infinite sets have the same number of members? These concerns revolved around the concept of infinity. In *Paradoxes of the Infinite* the concept of infinity was discussed in a way that prepared an environment for the developments to come. The question to be treated is: how was infinity treated in such an environment?

Bolzano’s life

Bolzano lived a tumultuous life as a mathematician and a political and religious radical. His radicalism led to his dismissal from the University of Prague and to the relative obscurity of his work to later generations. His work was rediscovered later in the nineteenth century; of this more will be said in later chapters. So complete was this obscurity that De Morgan independently discovered one-to-one correspondence in the intervening years. It is worth giving a brief

[88] I. Grattan-Guinness, “Bolzano, Cauchy and the ‘New Analysis’ of the Early Nineteenth Century” *Archive for History of Exact Sciences* 6, No. 5 (6.VIII.1970) : 372-400.

overview of Bolzano's dismissal since his absence from the cloisters of academia is the root cause of the long obscurity of his work, and its later discovery.

Bolzano took Catholic orders and occupied the chair of philosophy of religion at the University of Prague in 1805. Bolzano was removed from the chair in 1819. The dismissal was an outcome of comments Bolzano had made against war, social class, and rank while delivering church sermons. On the ecclesiastical level Bolzano was recognized as a Catholic in good standing throughout his life. But, the result of the dismissal was that Bolzano was never to hold any academic appointment. His dismissal was a matter of the Czech government, who had the power to remove him from academia. Bolzano split his remaining years between Prague, where he lived with his brother and the home of his benefactress outside of Prague.[89]

Bolzano's Work on Infinity

Bolzano's concern was with infinity. The definition of infinity Bolzano gives us is as follows. Infinite means that which cannot be counted, or what is equivalent, measured. When reading Bolzano it must be kept in mind that his terminology differs from that of the set theorists of the late nineteenth century. The term "countable" is one such example. "Countable" in *Paradoxes* means finite whereas later it would include the infinite set of cardinality $\aleph_0$. That is, an infinite series is one that has no last term. Bolzano writes: "I propose the name infinite multitude for one so constituted that every single finite multitude represents only a part of it."[90] An infinite series is not countable because it has no final member. As a result any number that could be counted to would comprise only part of infinity. Bolzano's "countable" can be paraphrased as "countable in full." When discussing an infinite set Bolzano writes: "Whence it

[89] Bolzano, *Paradoxes*, 10-15.
[90] Ibid, 79.

follows that the aggregate of all the above propositions enjoys a multiplicity surpassing every individual integer, and is therefore infinite."[91] Infinity then can be thought of as the multiplicity of which every finite multiplicity is part. Infinity can also be thought of as the multiplicity that is greater than every integer. Under Cantor this multiplicity will be referred to as omega. Bolzano comes closer to ordinals than this as will be discussed later.

Bolzano's discussion of points mimicked the later work of the mathematicians below. One area in which Bolzano foresees this work is when he writes of points; "a simple point of time or space has no such thing as a limit, but rather is itself the limit of an interval of time or of line."[92] This is similar to what Dedekind would write about real numbers. This characteristic of points will come up again in the discussion below of the Dedekind cut and the properties of the number line. In the discussion of limit points Bolzano also comes close also to an idea that the mathematician Bernhard Riemann had. Both men were concerned with the relationship between infinity and endlessness. Bolzano writes:

> No whit more satisfactory is the definition given by those who lean on the derivation of the word and say: the infinite is that *which has no end.* If they are thinking in this definition only of an end in time, only of cessation, then no other things could be infinite but those subject to temporal flux, whereas we also ask of things not so subject, like lines or abstract quantities, whether they be finite of infinite. But if they take the word end in a wider sense, say as equivalent with *limit* as such,[93]

Bolzano counts this restricted sense of "endless" as one of the erroneous definitions of infinity. A similar consideration was made by Riemann. The historian Morris Kline writes of Riemann:

> One of his (Riemann) novel thoughts was that we must distinguish between endlessness and infiniteness. For example, the equator of the Earth is endless but finite. In view of this distinction Riemann proposed an alternative to Euclid's axiom

[91] Ibid, 85.
[92] Ibid, 83.
[93] Ibid, 82.

> on the infiniteness of the straight line, namely, the axiom that all lines are finite in length but endless.[94]

Infiniteness and endlessness would find its way into much of nineteenth century mathematics. Considering the above it is perhaps not surprising that Riemann had influence on set theory through his work with manifolds. In the first part of the Bolzano quote, infinity is defined as "that which has no end" contrary to the distinction that Riemann made. Bolzano adeptly shows that this kind of infinity will not serve the purposes of the mathematicians and that end must redefined as limit. He eventually shows that this definition is also lacking. What is important here, though, is that both thinkers are engaging with an opposition between limit and end. In both mathematicians we see an engagement with limit, end, and infinity. The influence of *Paradoxes* had great breadth, paralleling, as it did, considerations in geometry.

Later, Bolzano advanced the concept of infinity with holomerism. Bolzano's definition of holomerism has within it part of the definition of infinity that Dedekind posited later. Holomerism, in Bolzano's own words, states two rules such that for an infinite set and one of its subsets:

> It is possible to couple each member of the first set with some member of the second in such wise that, on the one hand, no member of either set fails to occur in one of the couples; and on the other hand, not one of them occurs in two or more of the couples;[95]

This is Bolzano's first rule, which lays out one-to-one correspondence. In his second rule he writes:

> One of the two sets can comprise the other as a mere part of itself, in such wise that the multiplicities to which they are reduced, when we regard all their members as interchangeable individuals, can stand in the most varied relationships to one another.[96]

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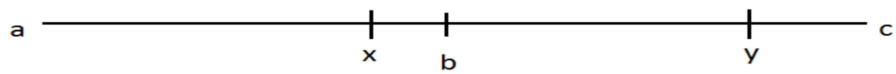
[94] Morris Kline, *Mathematics in Western Culture* (New York: oxford University Press, 1953), 422-423.
[95] Bolzano, *Paradoxes*, 96.
[96] Ibid.

Holomerism states that a one-to-one correspondence can be drawn between the members of the sets, and that one of the sets is only a part of the other. Bolzano gives the following example as an illustration of holomerism:



Bolzano tells the reader to imagine the line above with three points *a*, *b*, and *c* on it, such that *ac* is longer than *ab*. Suppose that two more points *x* and *y* are described along the line such that the ratio *ab*:*ac* = *ax*:*ay*. Whenever the position of *y* changes the position of *x* must change in order to maintain the ration of *ab*:*ac*. Thus, for any arbitrary point at which *y* is located, *x* is then located at a point such that a pair is created, such that *xy* is a unique pair of points and neither *x* nor *y* are members of any other pair created as *y* moves about the line. To put this another way, there is a one-to-one correspondence between every possible position of *y* and the resulting position of *x*. The first of Bolzano's rules has been fulfilled. One-to-one correspondence is the inheritance Bolzano bequeathed to later mathematicians. Dedekind's and Cantor's mathematics will turn on one-to-one correspondence. Bolzano then argued that *ax* was a part of *ay* since *ay* contained all the points *ax*+*xy*. Thus the second rule is also fulfilled. It is worth mentioning that this is an example in which Bolzano would say that *ax* and *ay* are different magnitudes of infinity and more will be said of this later.

It is worth discussing where Bolzano fell short of set theory. Donald Steele writes in his introduction to *Paradoxes*:

> In fact, as soon as we have said that Bolzano formulated holomerism more precisely than hitherto, viewed it more universally, and made it a symptom of infinitude sets, we have claimed as much as the *Paradoxien* warrant. Neither there nor in other writings does Bolzano exploit holomerism enough to be regarded as a precursor of Cantor in any greater measure than that.[97]

Steele argues that Bolzano was a forerunner of set theory only to a point. Holomerism and the one-to-one correspondence contained therein were not developed as far or as rigorously as set theorists would later. Specifically, Bolzano does not use one-to-one correspondence as criterion for two sets having the same number of members but only for detecting infinity. This is just what later mathematicians would use one-to-one correspondence for. One-to-one correspondence will become central to the work of Frege, Dedekind, and Cantor discussed below. Bolzano's holomerism was very close to Dedekind's later definition of infinity.

Bolzano also believed that calculations with infinity were possible. Bolzano tells us that these are justified because they are calculations with ratios not numbers. Bolzano writes:

> A correctly conducted calculation with the infinite is not a numerical determination of what is therein not numerically determinable (namely, not a numerical determination of the infinite multitude as such) but only aims at determining the ratio [or relationship (5)] between one infinite and another;[98]

Later Bolzano attacks the position that "When we pass to the whole infinite set of numbers, *n* simply becomes N_0."[99] N_0 cannot be considered the final number *n* in the set of natural numbers because there is no final term *n* that could be equivalent to N_0. While he cannot give the number that belongs to N, he can give the ratio between the set of natural numbers and the set of even natural numbers as $1:2=N_e:N$.[100] N here is close to what Cantor developed with ordinals, which is

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[97] Ibid, 25.
[98] Ibid, 107.
[99] Ibid, 108.
[100] The symbol ":" here denotes a ratio.

discussed below. Though not exactly since Bolzano's N is cumulative, as seen in the series below. Bolzano even goes so far as to write:

$$1^0N_0 + 2^0N_0 + 3^0N_0 + \ldots = N_0^2$$
$$1^0N_0^2 + 2^0N_0^2 + 3^0N_0^2 + \ldots = N_0^3$$ [101] [102]

Bolzano derives the second equation by multiplying the first by N_0. This process yields infinite degrees of infinity. Though less sophisticated than Cantor's ordinals, the germ of what Cantor accomplished is contained here.

Bolzano prefigures Cantor by arguing that there are different magnitudes of infinity. However, Bolzano deviated from cantor when he wrote: "On the contrary, many of them are greater (or smaller) than some other in the sense that the one includes the other as a part of itself (or stands to the other in the relation of part to whole)."[103] He also deviates from Cantor when he writes:

> What mathematician is there who, if he allows infinity of any kind, is not forced to concede that the length of a straight line bounded on one side but stretching to infinity on the other is infinitely great and nevertheless capable of being increased on the side hitherto limited?[104]

This allows for the difference in magnitude between the set of real numbers and the set of natural numbers since the natural numbers are a part of the set of real numbers. However, it is not restrictive enough. For it also allows for the difference between the set of natural numbers and the set of even natural numbers. This can be seen in the following example that Bolzano gave:

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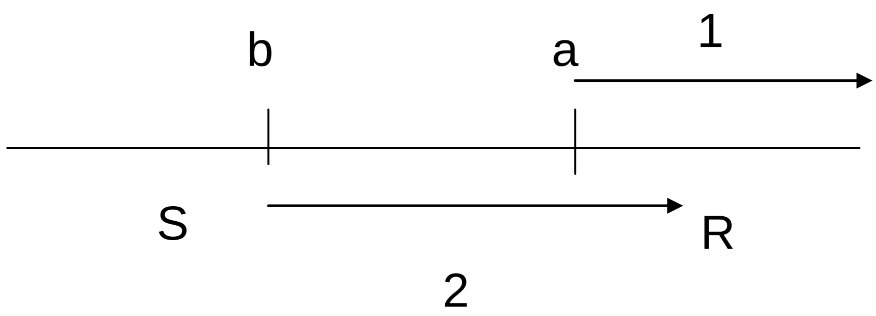
[101] Suppose we take the number 3^2. We can write this out as 3 x 3. We can further write this out as 3 + 3 + 3 = 9. If we next say 3 + 3 + 3 = 9 and want to know what 9^2 is then we write (3x9) + (3x9) + (3x9) = 9^2. This can then be written as 27 + 27 + 27 = 81 = 9^2. If you wanted to know what 9^3 is you could write $(3x9^2)+(3x9^2)+(3x9^2)= 9^3$ or (3x81)+(3x81)+(3x81)= 9^3. This is how these two series are generated.

[102] Bolzano, *Paradoxes*, 108.

[103] Ibid, 95.

[104] Ibid, 82.



Ray 1 begins at point *a* and travels to the right. Ray 2 begins at point *b* and also travels to the right. Bolzano says that ray 2 is longer than ray 1; however both lines are infinite. This is a conclusion that Cantor would reject. Bolzano also fails to give a rigorous definition of "part." It is clear that what he meant was a concept like the term "proper subset" later used by Dedekind. It may justly be asked at this point if the first rule of holomerism should have prevented Bolzano from stating different degrees of infinity. One-to-one correspondence could be applied to $N : N_e$ or to the points or measured feet of ray 2 and ray 1. As mentioned above Bolzano never meant the first rule of holomerism to be a justification for saying that two sets were of the same size. One-to-one correspondence is only used to tell if a set is infinite.

Bolzano states the nature of infinity. Next he set out to prove the existence of infinity. He does this by looking at assertions and assertions of assertions. For the purpose of this discussion, we will refer to these as recursive propositions. For example:

A

(A) is true

((A) is true) is true

(((A) is true) is true) is true

etc.

These propositions are only a part of the set of all propositions. The totality of propositions can be placed in one-to-one correspondence with these propositions since they are themselves propositions, as illustrated below:

A -- B

(A) is true --------------------------------- A

((A) is true) is true ------------------------ If B then C

(((A) is true) is true) is true -------------- (A) is true

((((A) is true) is true) is true) is true ----- B or A

For every proposition one selects, a recursive proposition can be matched to it. The two rules of holomerism are therefore fulfilled, and the set of all propositions is shown to be infinite. Bolzano writes:

> The aggregate of all these propositions, every one of which is related to its predecessor by having the latter for its own subject, and the latter's truth for its own assertion, comprises a set of members (each member a proposition) which is greater than any particular finite set. The reader does not need to be reminded of the similarity borne by this series together with its principle of construction to the series of numbers considered in S8.[105]

Thus Bolzano proves the infinitude of all propositions. It is interesting to note that this proof assumes something like the sixth axiom in Peano's *Principles of Arithmetic,* which concerns succession in natural numbers. This axiom will be discussed in detail in a later chapter.[106] This is important because it suggests common thinking between the two strains of foundations, Peano belonging to the logic strain and Bolzano as the beginning of the set theoretical strain.

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[105] Ibid, 85.
[106] A short note on this, Peano's sixth axiom, as we will see, states that if *a* is a natural number then *a*+1 is a natural number.
Likewise we can say something like, if *a* is a proposition then we can say that the assertion of *a*, *a* is true, is also a proposition.

Bolzano qualifies the above proof by saying that it concerns “non-actual members.”[107] In a sense Bolzano admitted his own cheap trick and gives a more substantive proof of the existence of infinity. In *Paradoxes of the Infinite* Bolzano wanted to posit and solve the paradoxes of infinity. A few words should be said here on what a paradox is in mathematics. A paradox in this context is when the mathematician’s intuition comes into conflict with some new application of previously accepted mathematics. This occurs when accepted canon leads to absurdities. The first paradox Bolzano treated was the infinity of the natural numbers. Bolzano stated the paradox as follows: “If each number . . . is by definition a merely finite set, how can the set of all numbers be infinite?.”[108] In other words, if any natural number *n* and every natural number that precedes *n* form a finite set then how can the set of all natural numbers be infinite? With whatever set we choose wouldn’t we have a finite number *n* of members? Bolzano’s solution to this paradox is to point out that in the set of all natural numbers there is no *n* that occupies the last place, as stated in his definition of infinity as not having a last member.[109]

Bolzano engages with the ancient conundrum known as Zeno’s paradox.[110] This paradox is stated through the series 1 + ½ + ¼ + 1/8 + 1/16 . . . this series continues on to an infinite number of terms. Each of these terms is a finite quantity like with the set of natural numbers. Therefore it seems that the sum that is described by this series should be infinite. On the other hand it seems that the series tends towards a limit of 2. Bolzano solves the paradox as follows:

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[107] Bolzano, *Paradoxes*, 90.
[108] Ibid.
[109] This discuss ion is contained in Bolzano, *Paradoxes*, 90-91.
[110] The paradox is usually told in the following way. The ancient Greek warrior Achilles is to race a tortoise. The rules of the race stipulate that the tortoise is to have a head start. Now, suppose the tortoise travels one foot per second and Achilles travels two feet per second. Also, suppose that the tortoise has a one second head start. When the tortoise reaches one foot Achilles will start running. When Achilles reaches one foot the tortoise will have traveled another half a foot. When Achilles travels half a foot the tortoise will have traveled a quarter of a foot. The paradox is that it seems as though Achilles will never catch the tortoise.

> The semblance of a paradox which some may see in it originates in forgetfulness of the fact that the addends become smaller and smaller. Nobody can be surprised if a sum of addends each of which halves its predecessor can never surpass the double of the initial addend, since at however late a term in the series we halt, exactly so much is wanting to make up that double, as the term in question has value.[111]

Addends here are the terms in the series. The sum of the members of the set of natural numbers is infinite because the quantities become larger. Even the sum of the terms of the series 1+1+1+1+1 . . . is infinite since the quantities do not change. In Zeno's series the quantities become smaller at a certain rate.[112] This is why its sum is finite. What Bolzano is pointing out is that no matter what term we take, the sum of the terms will be exactly that far from $2n_1$, that is twice the first term. This is exactly the value that the next term will be half of. So, if we take the first three terms of the above series the sum is 1¾ which is ¼ away from $2n_1$ which is 2. Thus, Bolzano solved the original paradox of infinity.

Bolzano considers what he calls the paradox of continuous extension. This deals with an apparent paradox in the idea of a continuum. Bolzano tells us that this paradox came from considerations of time. The example of time is particularly useful in this explanation because time can be thought of as a line. An analog can be seen in the solution of this paradox with the investigations of Richard Dedekind that are covered below. The paradox is given in two parts. The first part asks that if something is extended, how can it arise from parts that are not extended? That is, how can a line be one dimensional yet constructed completely out of dimensionless points, even if there are an infinity of them? Where does the other dimension come from? Bolzano meets this paradox by stating that the points of the number line do not have to be one dimensional in order for the line to be one dimensional. More generally, Bolzano states

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[111] Bolzano, *Paradoxes*, 95.

[112] It is worth pointing out that Zeno's series is not finite just because the addendums become smaller. For example, the series 1/1, ½, 1/3, ¼, … the addendums sum to infinity.

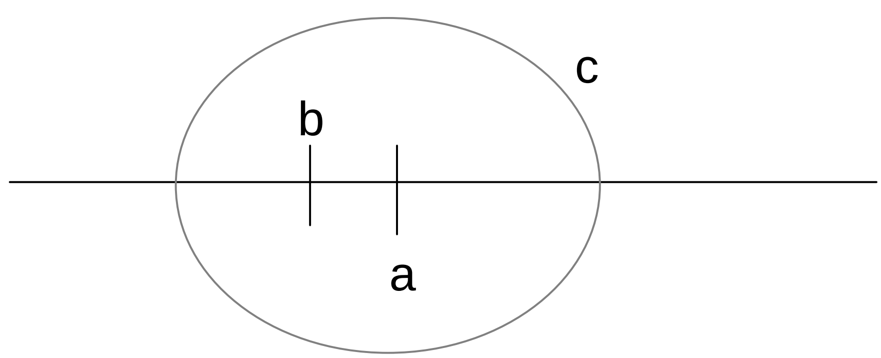
that it is not necessary that characteristics of the whole must be found in its parts. Bolzano uses the example of the automatons of his day:

> An automaton has the property of almost deceptively mimicking the movements of a living person, whereas its separate parts, its springs, its little wheels and the like, do not possess any such property.[113]

The parts of the automaton give rise to human-like mimicry without being human themselves. The second part of the paradox states another attack on the continuum. Bolzano states this attack as follows: "That every two instants or points or substances still stand at a certain distance apart and hence fail to constitute a *continuum*."[114] Bolzano tackles this part of the paradox by defining continuum in a precise way. Bolzano defines "continuum" in the following way:

> we are forced to declare that a continuum is present when, and only when, we have an aggregate of simple entities (instants or points or substances) so arranged that each individual member of the aggregate has, at each individual and sufficiently small distance from itself, at least one other member of the aggregate for a neighbor.[115]

To understand this definition and what it accomplishes, a definition of neighbor and neighborhood is necessary, the two terms would have been familiar from Bolzano's work in analysis. A neighbor is a member of a set occupying a position within a vicinity of an arbitrary size around another member of the set. A neighborhood is this vicinity. For example:



[113] Bolzano, *Paradoxes*, 128.
[114] Ibid.
[115] Ibid, 129.

c is the neighborhood around point *a*. *b* is the neighbor to *a* within the neighborhood *c*. Bolzano's definition of continuum states that for any point *a*, a neighborhood *c*, no matter how small, can be described around point *a* such that there is a neighbor *b* within the neighborhood. This is what a continuum is for Bolzano. Before it was argued that in a so-called continuum any two points, no matter how close together they were, still had a distance between them, therefore there was no continuum. Bolzano argued that, in fact, in a continuum every point can have described around it an arbitrarily small neighborhood such a neighbor can be found to fall within it.[116]

Bolzano discussed a paradox that originated with Galileo. This paradox states that "the circumference of a circle is equal in magnitude to its center"[117] That is, the area of a point at the center of a circle is equal to the area of the circumference of that circle. This, of course, is nonsense; neither a zero dimensional point nor a one dimensional circumference has any area at all. This paradox stems from a misunderstanding of infinitesimals.[118] The paradox states that the length of line *pn* is the radius of a circle with an area equal to the area of a ring formed between two circles *pm* and *pr*. This is readily accepted by Bolzano. The smaller circle has the radius of *pm* and the larger the radius of *pr*.[119] As the line *pr* moves toward the line *ab* the point *m* moves along the curve toward *b* and the point *n* moves along the diagonal toward *a*. Hence, as *pr* moves toward *ab* both areas become smaller. The paradox states that the circle *pn* shrinks to an infinitesimal area and the ring *mr* to an infinitesimal ring. The circle and the ring shrink down to a point and a circumference. Hence, the point *p* that *pn* has shrunk to and the circumference around *p* that *mr* has shrunk to are the same size. It is counterintuitive to think of a point at the

[116] This solution finds its way into Phillip Jourdain's introduction to the English edition of Cantor's *Contributions*.
[117] Bolzano, *Paradoxes*, 141.
[118] See illustration in appendix. This illustration is from: Bolzano, *Paradoxes*, 141.
[119] See the illustration in the appendix.

center of a circle and the circumference of that circle as having the same area; they do not have an area. This was Bolzano's solution. He stated that *pn* disappears into *p* and that it cannot be said that *p* is the center of the circle any longer. *mr* disappears into a circumference around *p* and it no longer makes sense to talk about *mr* being a ring between pm and *pr*. Bolzano states the fallacy in two parts, the first which pertains to the radius of *pn*, as believing that: "When our attention is transferred from *p* to *a* and no such radius *pn* exists any longer, the point *a* still survives as the midpoint of the ultimate circle *pn*."[120] The second part pertains to the ring, he states:

> The ring left between with the smaller radius pm and that with the larger radius pr 'ultimately' becomes the circumference of the previously greater circle, 'ultimately' meaning: when the two radii and hence the two circles have become equal.[121]

Bolzano stated the following equation: "$\pi.pn^2=\pi.pr^2-\pi.pm^2$."[122] For Bolzano, the equation shows that *pn* and the ring between *pm* and *pr* just do not exist, not that their areas are equal. Thus, Bolzano resolves the apparent paradox. The cause of the preceding paradox is the misunderstanding of the infinitesimal. It is helpful to delve into this misunderstanding. It is composed of this, that, when *pr* reaches *ab* the areas concerned are zero. This creates the illusion that these geometric entities still exist, that they have just such a measurement. This came about because as *pr* becomes an infinitesimal distance from *ab* the areas become infinitesimal. The errant mathematician takes this as license to say that since the equivalence works for infinitesimals it holds for zero. But that is a philosophical move not a mathematical move, and an errant move according to Bolzano.

[120] Bolzano, *Paradoxes*, 142.
[121] Ibid.
[122] Ibid.

Bolzano also defends the idea of the infinitesimal against mathematicians who argued that the infinitesimal does not exist. Raymond Wilder quotes the Swiss mathematician Johan Bernouli as saying: " A quantity, which is increased or decreased infinitely little, is neither increased nor decreased."[123] Bolzano states: "A number of mathematicians sought to avoid such contradictions by taking refuge in the declaration that infinitely small quantities are in reality *mere zeros*."[124] These mathematicians argue that an infinitesimal that was added or subtracted from a lower order infinitesimal, a finite quantity, or any infinite quantity; should be treated as zero. For example, $2 \pm m = 2$. m here is represents an infinitesimal quantity. Bolzano argues that if we create an irrational fraction such as M/N and we say of a rational number p/q that $p/q < M/N < (p+1)/q$ then we may replace M with $M \pm m$ with no difference to the truth of this formula no matter what p or q may be. However, if M/N is rational then there is a rational number p/q such that $M/N = p/q$. In this case M/N forms a Dedekind cut, these cuts are discussed below. It is important to recognize the relevance of this particular paradox to Dedekind's work. In this situation the mathematician can no longer replace M with $M \pm m$ and maintain the truth of the equation $M/N = p/q$. For if $M/N = p/q$ then either $(M \pm m)/N < p/q$ or $(M \pm m)/N > p/q$.[125] Thus Bolzano gives a circumstance in which there is a very important difference between zero and infinitesimal.

Bolzano's Logic

Although this chapter in primarily concerned with Bolzano's views on infinity there should be a brief mention of his work in logic. Most of Bolzano's logical work is contained in

[123] Wilder, *Introduction*, 189.
[124] Ibid, 120.
[125] I will not go into the Dedekind cut here the reader should return here after reading the chapter on Dedekind. What will be recognized here is that the difference between M/N and $M \pm m/N$ can be the difference between being a member of A_1 or A_2 when a cut is made at p/q.

Wissenschaftslehre. Steele argues that Bolzano deemphasizes the importance of psychology in logic.[126] In this, Bolzano agrees with Frege's aim to remove psychology from logic which the latter states in the *Begriffschrift*. Steele mentions especially Bolzano's concept of "*satze an sich*" contained in *The Wissenschaftslehre*.[127] This term refers to an absolute proposition similar to Frege's concept of the *Begriffschrift*, or "concept writing," which will be described below. *Paradoxes of the Infinite* is the more relevant work to the present investigation. This chapter is concerned with Bolzano as a mathematician more than as a logician. Steele observes: "Apart from references by Hamilton and Venn to matters of smaller moment, the logicians took longer to discover Bolzano than did the mathematicians."[128] As a mathematician, Bolzano's impact was probably greater since his ideas were discovered there earlier. Still there is something to be said here of Bolzano's logical work. This work was applauded in due time as when Heinrich Scholz writes that: "With Frege, he (Bolzano) is one of the greatest two formal logicians in the German literature of the nineteenth century."[129] The lack of attention to his logic has been remedied and recently his work has come out in English translation. Bolzano forms a link between the set theory strain of foundations and the mathematical logic strain. His books *Paradoxes of the Infinite* and *Wissenschaftslehre* are his great contributions to the logical and set theoretical. The division between logic and set theory is not absolute in these two books. Bolzano calls, for example, for a quasi-axiomatic treatment of infinity in *Wissenschaftslehre*. He calls this the second rule for the conveyance of scientific knowledge.[130] Bolzano formulated these rules for the guidance of science. This treatment is similar to that of Giuseppe Peano discussed below. In this

[126] Bolzano, *Paradoxes*, 41.
[127] Ibid, 45.
[128] Ibid, 48.
[129] Ibid.
[130] Ibid, 47.

we see the influence of Leibniz who also stated the importance of axioms. Bolzano was heavily influenced by this giant of seventeenth century logic. A good chunk of *Paradoxes* is devoted to metaphysics, a field that at the time would have been still dominated by Leibniz's *Monadology* and *Discourse on Metaphysics*.

Conclusion

Bolzano's book *Paradoxes of the Infinite* forms the beginning of what became set theory. Bolzano also forms a link between logic and set theory in *Paradoxes* and *Wissenschaftslehre*. In *Paradoxes*, he sets out the issues that later set theorists would deal with. The paradoxes that Bolzano attempted to solve called for new methods which opened avenues of research for future mathematicians. But even more than this, *Paradoxes* represented a new way to think about infinity that would later become its own field in mathematics. It was the rediscovery of Bolzano by mathematicians at the University of Berlin that would give Bolzano's ideas new life among the first generation of set theorists.

4: The Youth of Mathematics

> Historians of logic usually trace back the official birth of the contemporary mathematical form of logic to the first edition of George Boole's *The Mathematical Analysis of Logic* of 1847. It has become common, however, to speak of a second birth, due to the publication of Gottlob Frege's *Begriffschrift* in 1879.
> -Massimo Mugnai[131]

> Logic is the youth of mathematics and mathematics is the manhood of logic.
> -Bertrand Russell[132]

Introduction

Gottlob Frege will be central to the remainder of this study. He ushered in a second founding of mathematical logic. The historian Akihiro Kanamori writes that Frege was the "greatest philosopher of logic since Aristotle."[133] Frege's *Begriffschrift*, by translation, is a conceptual notation and took the mathematical program in logic further. Frege is also an early founder of logicism which was an influential philosophy to the mathematicians who followed. Frege is the first of two logicists discussed, the other will be Richard Dedekind. I will also discuss Giuseppe Peano who, while he was influential to formalism, can be counted as in the same school as Frege and Dedekind. Frege set the program for logicism. He did this by criticizing and then amending prevailing views of number and arithmetic. One such example was Frege's attack on psychologism. I will argue that this is one of the big moves in the history of logic, and show how Frege attacked some of the figures already discussed. Contemporary

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131 Mugnai, "Logic and Mathematics", 311.
132 Bertrand Russell, *Introduction to Mathematical Philosophy* (2010) University of Massachusetts, http://people.umass.edu/klement/imp/imp.html (accessed 7/14), 194.
133 Kanamori, "Empty Set", 274.

historians, such as Massimo Mugnai, consider Frege to be just as instrumental in the creation of mathematical logic as George Boole was. This is a view I agree with and it is in large part because of Frege's attack on psychologism. No one did more in the nineteenth century than Gottlob Frege to found logicism. Our discussion of Frege will focus on two works, *Begriffschrift* (1879) and *The Foundations of Arithmetic* (1884). Frege was born in 1848, the very year that continental Europe exploded in revolution.[134] He was raised in the Lutheran faith common to German children of the time. He attended the University of Jena and the University of Gottingen, where he received his doctorate in 1873. He then returned to the University of Jena where he became a lecturer. He would spend the rest of his career at Jena.

Logic Moves to the Continent

Frege and the mathematicians he collaborated with were all continental, and for the most part German. To determine why this shift occurred the historian must reach outside of the world of mathematics. I have mentioned that the center of mathematics in general had moved from France to Germany. Logic, in particular, had moved from Britain to Germany. One of these mathematicians, Peano, was not German. We may use this to guess at a cause for the shift in dominance to Germany. Both Germany and Italy had become nations and thus completed the process begun by the failed revolutions of 1848. While there was still uncertainty, the advantage of domestic tranquility mentioned by Bruun earlier was no longer unique to Britain. Germany had lagged behind Britain in the Industrial Revolution and so also, according to Michael Shenefelt and Heidi White, in the analogy that was available to the logicians of Britain. The historian W.O. Henderson states that this is because Prussia, the largest kingdom of Germany

[134] This biographical account is largely taken from the *Stanford Encyclopedia of Philosophy*.

before unification, lacked money and fought wars early in the nineteenth century.[135] Geoffrey Bruun writes of Germany's ascension: "Great Britain had been the 'workshop of the world' for a hundred years, but by 1900 Germany and the United States had cut down the British lead."[136] Bruun goes on to write of the export of machinery (an economic indicator): "By 1880 the order of precedence was Great Britain, Germany, the United States."[137] Germany had overcome its economic deficiencies. It had become an economic power in the world. The industrial analogy that had made Britain so dominant in the field of logic had come to Germany. This is just what we see. The second half of this narrative is dominated by Germany. Shenefelt and White's theory seems to be correct. If we look at the time table logic does seem to have followed the Industrial Revolution.

The *Begriffschrift*

The *Begriffschrift* contains Frege's logical system. Frege meant his notation to be a language for science. One can see in it many examples from science when Frege translates his notation. Frege advises the reader: "When a problem appears to be unsolvable in its full generality, one should temporarily restrict it."[138] Frege's notation was just one such restriction for use by science and mathematics. Frege did not believe that his notation was a ratiocination. However, he did believe that it could lead to one in the future. Later, in 1882, Frege would write of his program in logic that:

> My intention was not to represent an abstract logic in formulas, but to express a content through written signs in a more precise and clear way than it is possible to do through

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135 W.O. Henderson, *The State and the Industrial Revolution in Prussia, 1740-1870* (Liverpool, UK: Liverpool University Press, 1958), 190.
136 Bruun, *Nineteenth Century*, 151.
137 Ibid, 152.
138 Gottlob Frege, "Begriffschrift" in *From Frege to Godel* (Cambridge, Massachusetts: Harvard University Press, 1967), 6-7.

> words. In fact, what I wanted to create was not a mere calculus ratiocinator but a lingua characterica in Leibniz's sense.[139]

In the field of logic Frege attempted to provide an improvement upon how concepts were communicated. Frege wanted a way to speak precisely in science, and in the *Begriffschrift* many of his examples are from science. He was not interested, as Boole was, in an algebra of logic. Frege wanted a more fundamental and articulable language for science. Frege compares his logic to a microscope:

> The microscope, on the other hand, is perfectly suited to precisely such goals, but that is just why it is useless for all others. This ideography, likewise, is a device invented for certain scientific purposes, and one must not condemn it because it is not suited to others.[140]

With this new language Frege felt he could have the precision necessary to begin constructing a foundation for mathematics and thus science. The difference lay in this. When Boole writes $ab=a$ this imparts information. However, it is not very useful information for science. Suppose the mathematician states "for any number $n > 2$, if it is prime, then $n+1$ is a multiple of two," referring to this as "the class of primes, is contained in the class of numbers that if you add 1 you will have multiple of two" is woefully cumbersome. There is a feeling that what the mathematician wants to talk about is numbers, not classes.

Frege replaces logic's subject/predicate distinction with the argument and the function. Frege gives the following example:

> A distinction between subject and predicate does not occur in my way of representing a judgment. In order to justify this I remark that the contents of two judgments may differ in two ways: either the consequences derivable from the first, when it is combined with certain other judgments, always follow also from the second, when it is combined with

[139] Ibid, 2.
[140] Ibid, 6.

> these same judgments, (and conversely,) or this is not the case. The two propositions "The Greeks defeated the Persians at Plataea" and "The Persians were defeated by the Greeks at Plataea" differ in the first way.[141]

For Frege two propositions may differ so that either the other is deduced from the first or not. The subject switches between the Greeks and Persians. These two propositions differ in the first way, and each is derivable from the other. And so, for Frege, they differ in an uninteresting way and in symbolization they are just equivalent. So, subject and predicate are irrelevant in symbolization. This is an important point of departure from Boole's notation. As we recall from the quantifier debate, the subject and predicate distinction had hitherto been of great importance to logicians. Boole symbolized propositions about classes, whereas Frege symbolized concepts. Concepts here should not be taken as mental objects; concepts are from outside the human mind. In particularly, Frege symbolized the content a proposition about classes had. What can be said in Frege's notation is said of individuals. Frege calls his system a *Characteristica Lingua* in the fashion of Leibniz. It is worth delving into what this means. N.I. Styazkhin writes of the C*haracteristica Lingua*:

> Here Leibniz developed one of his favorite ideas: "an alphabet of human thought that makes it possible to deductively derive new ideas by means of definite rules for combining symbols." Here the logical idea of pasigraphy is clearly distinguished from the linguistic idea of creating a "universal language."[142]

The important point here is the ability to "deductively derive new ideas by means of definite rules for combining symbols." Logic then is not a clarification of ideas in mathematics; rather, the *Begriffschrift* is a generating engine of scientific discoveries. Frege begins the *Begriffschrift* by outlining the most fundamental part of his notation. Frege introduces two symbols:

[141] Ibid, 12.
[142] Styazhkin, *History of Mathematical Logic*, 65.

$$\text{―――――――}$$

$$\vdash\text{―――――――}$$

The top symbol denotes the taking of a formula "paraphrastically," in Frege's words. When this symbol precedes a formula it is understood that we are not actually asserting anything. In the *Begriffschrift* this symbol is rarely used. Frege writes of this symbol that:

> We must be able to express a thought without affirming it is true. If we want to characterize a thought as false, we must first express it without affirming it, then negate it, and affirm as true the thought thus obtained.[143]

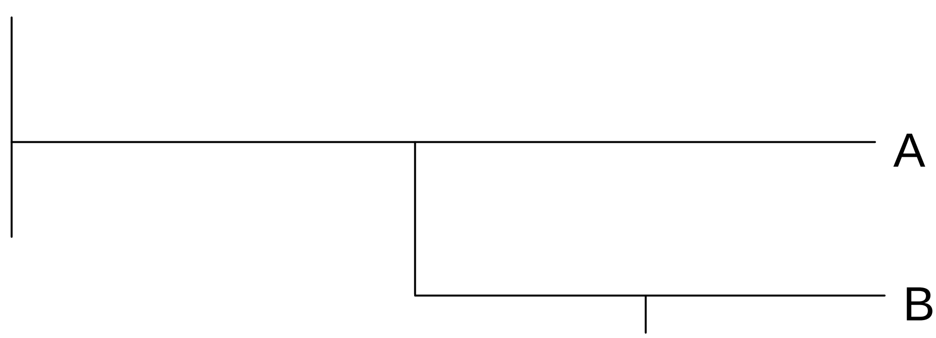
This symbol then gives us an environment in which we can prepare statements; for example the negation of which will be asserted. The symbol on the bottom has appended to it the vertical "judgment" stroke. When this precedes a formula, we understand that the formula is being asserted. With this stroke the formula gains a truth-content and we may begin to speak meaningfully about what that content entails. Frege gives us the example that the assertion that opposite magnetic poles attract can be taken "paraphrastically" as, opposite magnetic poles attracting.[144]

Frege introduced new ways of symbolizing negation. This was closer to the modern conception of negation in that it did not refer to the complement of a class, as it does in Boole's system. Frege's system is not an algebra of classes and so in negation, what is asserted is the proposition negated with no recourse to classes. Frege also atomized negation. Not-a is then not a single entity, but rather it is an entity negated. Negation is separated from what it negates in a

[143] Frege, "Begriffschrift", 11.
[144] Ibid.

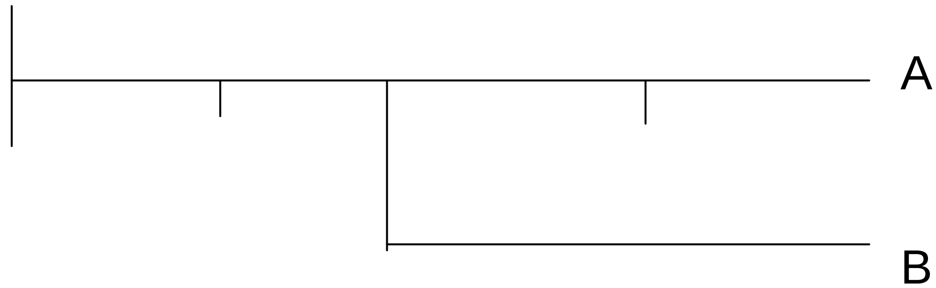
way that is impossible when, for instance, Boole writes (1-a). There is a good reason for Frege to want a paraphrastic environment available to him. In Frege's negation we have to be able to say that if a thing is negated, then that thing itself can be asserted. The paraphrastic environment is a staging ground that allows us to do this.

The logical operations Frege uses are implication and negation. All the traditional copulas are expressible in his system. Frege addresses both forms of disjunction. He writes: "Of the two ways in which the expression "A or B" is used, the first, which does not exclude the coexistence of A and B, is the more important, and we shall use the word "or" in this sense."[145] Where Boole would write A + B Frege would write if not-A then B, this expresses something deep and entirely novel about disjunction. These situations are equivalent. Frege's disjunction is inclusive, though the question of inclusive or exclusive disjunction loses relevance in Frege's notation. Frege handles disjunction in his notation as follows:



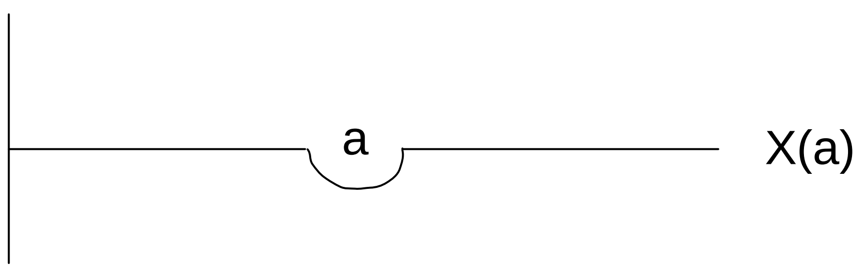
And conjunction:

[145] Ibid, 19.

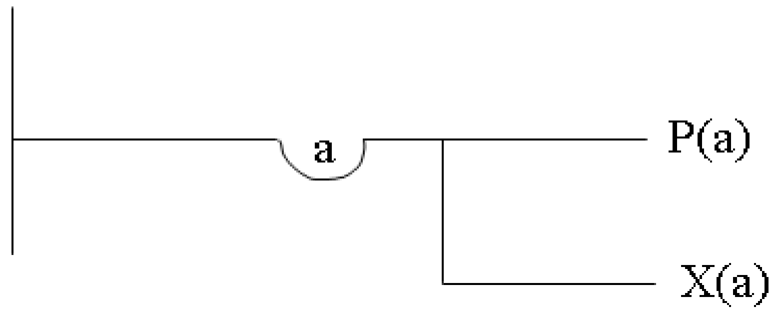


Frege handles conjunction in the above manner. One need only use a truth table to see that this is equivalent to “A and B.” It would be useful to go ahead and here give a further explication of Frege’s system. In the above illustration we first notice that the judgment stroke is present. There is a short downward mark to the right of the judgment stroke; this is the negation mark. The implication begins with the lower letter B as the antecedent. The top letter A is the consequent. Hence the above notation reads, it is not the case that if B then not A. This is logically equivalent to A and B.

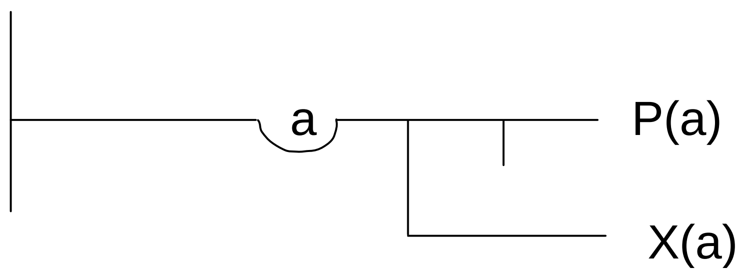
As mentioned above. Frege allows for the handling of the universal quantifier in the following way.



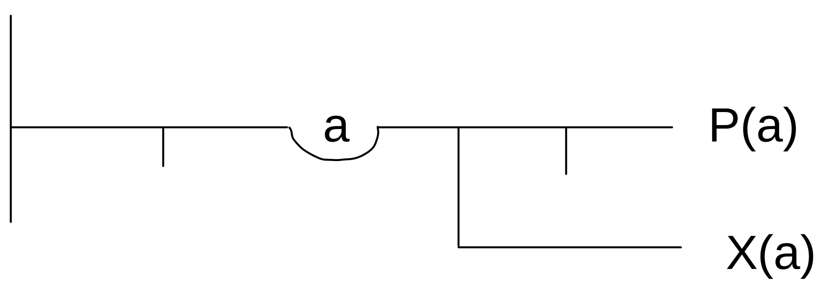
Frege states that an *a* placed in a “concavity,” as above represents a generalization. The above notation means for all *a Xa*. This allows Frege to symbolize the logic square. He represents A as:



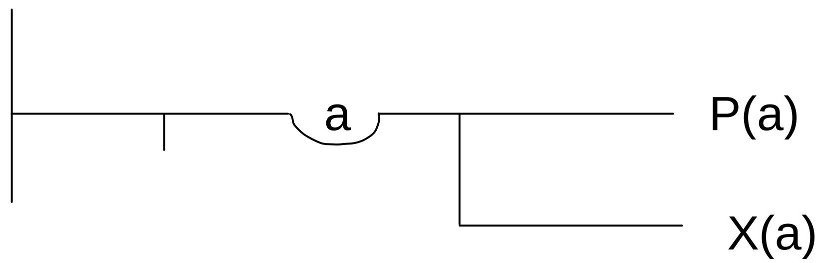
He represents E as:



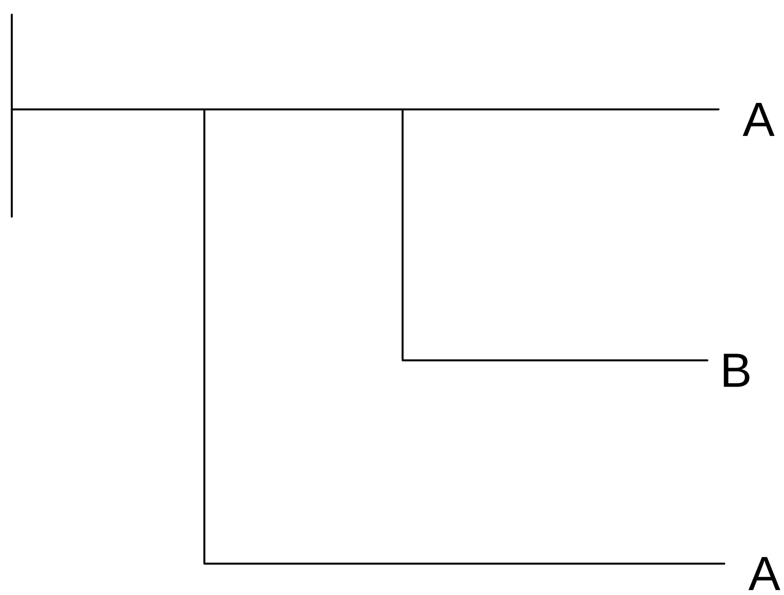
He represents I as:



He represents O as:

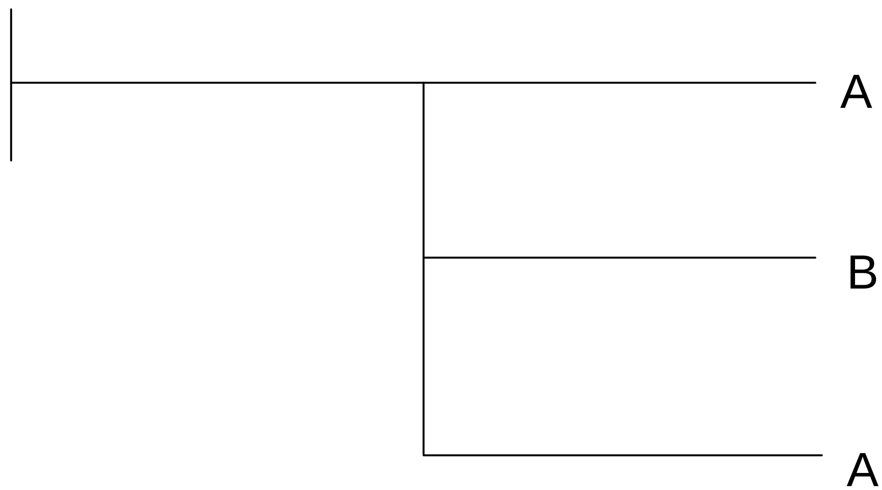


The following situation is translated as the familiar axiom “if A then, if B then A”:



It is worth noting that the notation above could be translated as: it is not the case that, A and not, if b then A.[146] This is an equivalence between implication and conjunction.[147] Occasionally Frege will write a proposition of the form:

[146] In modern notation: $\sim (A \wedge \sim (B \rightarrow A))$.
[147] This equivalence is derived by double negation.



Here, the bottom A and B form an application which in turn implies A. In modern notation, this proposition would be: $(A \rightarrow B) \rightarrow A$.[148]

Frege has a discussion of what a function is. Frege asks the question: in the proposition "The Greeks defeated the Persians at Plataea" is "defeating the Persians" the function, or is "being defeated by the Greeks" the function? Frege's answer is that it depends whether we take Persians or Greeks to be the argument, the replaceable part of the proposition. In this, the logician is at liberty. Frege writes:

> The situation is the same for the proposition that Cato killed Cato. If we here think of "Cato" as replaceable at its first occurrence, "to kill Cato" is the function; if we think of "Cato" as replaceable at its second occurrence, "to be killed by Cato" is the function; if finally, we think of "Cato" as replaceable at both occurrences, "to kill oneself" is the function.[149]

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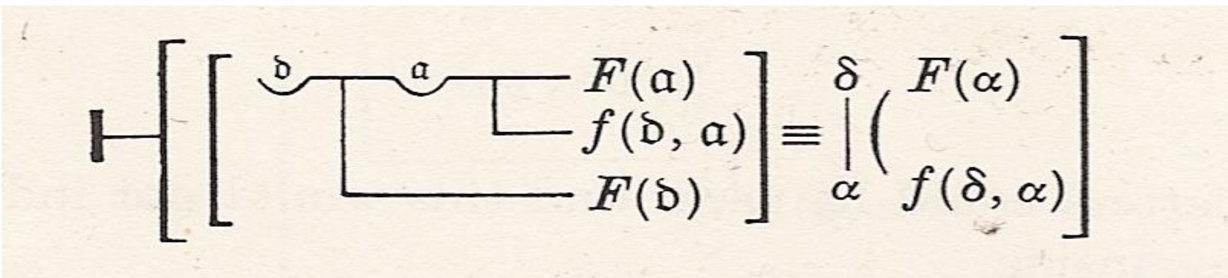
[148] This will be the end of the explication of Frege's notation. See the appendix for a continuation of this discussion.
[149] Frege, "Begriffschrift", 22.

Frege can express all three of these situations as: *f(a), f'(b), f''(a, b),* and even *f'''(b, a)*. In these cases, the function changes with the arguments and their order. In general, Frege will use two arguments in a function as he does when discussing sequences. In the *Begriffschrift*, Frege attempts an explication and symbolization of sequences. Frege begins with the following equivalence:[150][151]



The first equivalence states that if all *d* have the property *F*; then, if for all *a*, *a* is the output of the function *f* performed on *d*; then all *a* have the property *F*. This can be shortened to the proposition, "the property *F* is hereditary in the *f* sequence." The natural numbers give us a good example of what Frege is up to here. Suppose we define *F* as the property of being a natural number. Now suppose *d* has property *F*, and suppose that for any *a*, *a* is the result of some function *f* on *d*. Let us take this function as *f(d, a)*, as *d*+2=*a*. If this is true then *a* has the property of being a natural number. Therefore, the property of being a natural number is hereditary in the sequence formed by adding two to each natural number. This is because the output is also a natural number. It is important to remember that Frege is only setting a convention for symbolizing. He is not asserting anything but that:[152]

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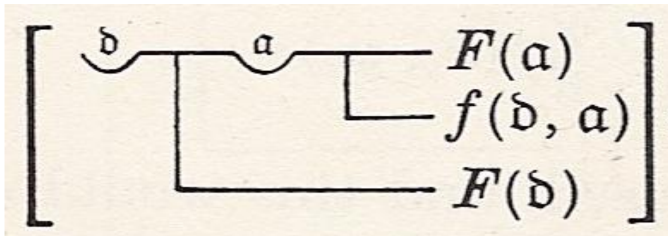
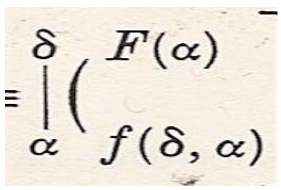
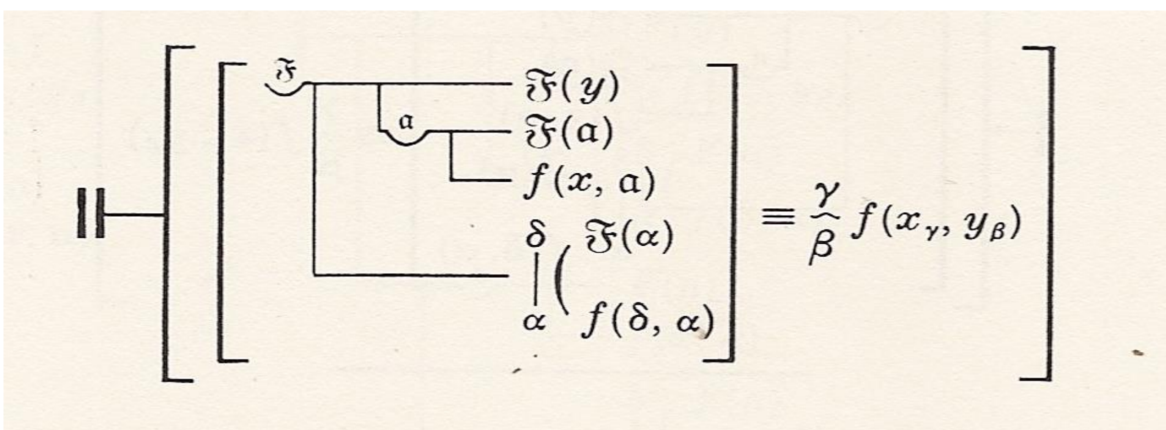
[150] It should be mentioned here that Frege does not give use an existential quantifier. This is only a matter of convenience since we can existentially quantify a proposition with a universal quantifier, as in:$\sim(x) \sim P_x$.
[151] Frege, "Begriffschrift", 55.
[152] Idib.



Will from now on be symbolized as this:[153]



Frege states a second equivalence:[154]



[153] Idib.
[154] Idib, 59.

Frege writes of this equivalence:

> If from the two propositions that every result of an application of the procedure f to x has the property F and that property F is hereditary in the f-sequence, it can be inferred, whatever F may be, that y has property F, then I say: "y follows x in the f-sequence", or "x precedes y in the f-sequence."[155]

It would be best to take this very tricky argument in three steps:

> 1. F is hereditary in the f-sequence.
>
> 2. If *a* is the result of *f* performed on *x* then *a* has property F.
>
> 3. If 1 and 2 force *y* to have the property F, it can only be because *y* follows *x* in the f-sequence. This is because if *y* appeared before *x* in the f-sequence then 1 and 2 would not imply that *y* had F. they would have just nothing to say about *y*. It is that they do imply, for this relationship to exist between 1 and 2, and *y* having the property F, *y* must follow *x*.

Let us look at an example to parse out what Frege is saying here. Suppose F is the property of being an even number. Suppose *f* is the function 2n. Suppose *x* = 5. The f-sequence is the sequence of even numbers and it is obvious that F is hereditary in it. Suppose *a* is the outcome of performing *f* on *x*, so *a* = 10. If *y* is some unknown quantity such that we do know that it is even, that it has property F, then we can only know that by knowing that *y* follows *x* in the f-sequence.

Frege's Philosophy of Mathematics

In the *Begriffschrift* Frege was not concerned with representing language as a part of mathematics. One can hardly look at the equivalencies just mentioned and say that they are mimicking human language. They seem to be grasping at a deeper conceptual truth. Nor was he concerned with the syllogism. There is nary a mention of the syllogism in the *Begriffschrift.* Frege also railed against the psychologism in the logic of his day. Boole's 1854 opus was titled *Laws of Thought*. We should take a moment here to define exactly what we are to mean by

[155] Frege, "Begriffschrift", 60.

psychologism. Psychologism is the view that logic comes from the mind and not from the world, that it is a description of correct mental function and nothing more. This view tends to emphasize language because in this view language and logic come from the same place. Much of the logic in the first chapter will be criticized by Frege as psychologistic. Frege's criticism of this logic is one of the big moves in the history of logic. The historian Nicla Vassallo argues that if Boole is psychologistic then so was Frege. She argues that, on the subject of where these laws existed, both logicians gave the same answer.[156] Frege states that all thought is essentially the same. Thus, there must be a set of rules for all thought. I think that first, Boole's system is more of an attempt to correct human thinking than Frege's, because of historical currents I have already mentioned. Boole's system is prescriptive. Second, as mentioned, Frege means his notation to be a *Lingua Characteristica*, and as N.I. Styazkhin tells us, this means logic must not just be language in mathematical form. For Frege the project was to generate new scientific knowledge by new combinations of symbols. Suppose we know that two chemicals mixed together in water form an alkali solution; suppose further that we know that if a solution is alkali then it conducts electricity. Then, once we have proven that the two chemicals mixed together form an alkali solution then we can surmise that mixing the two chemicals together forms a solution which conducts electricity. This is really just the transitive property. Of course scientists do this sort of thing all the time. However, if the chain became longer by several links the scientist would likely feel nervousness at not testing the links with experiment. Frege, and we will see that to a lesser extent Peano, hoped science could reason to such a long chain. The fact is Frege refers to something called psychology in logic, and Frege looks very negatively upon it. Boole on the

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[156] Nicla Vassallo, "Psychologism in Logic: Some Similarities between Boole and Frege" in *A Boole Anthology: Recent and Classical Studies in the Logic of George Boole* (Dordrecht: Kluwer Academic Publishers, 2000), 311-326.

other hand is largely silent on all of this; there is no talk of generating new theories in science. Vassalo goes on to quote Frege:

> Neither logic nor mathematics has the task of investigating minds and contents of consciousness owned by individual men. Their task could perhaps be represented rather as the investigation of mind; of the mind, not of minds.[157]

What Frege wanted to symbolize was concepts behind arguments. For example, while Boole was happy enough to symbolize "all *a* are *b*" as *"ab=a"* Frege went a step further and wrote "for all *things*, if it is *a* then it is *b*." Frege's logic was propositional. What is lurking behind "all *a* are *b*" is just that "if something is *a* then it is *b*." This was the birth of mathematical logic, a birth that was begun by Boole.

The philosopher Dale Jacquette writes of Frege's Platonism:

> Frege adopts a Platonic theory of Gedanken or propositions. The concept is that of abstract meanings of concrete sentences. It is similar to and possibly derived from Bernard Bolzano's doctrine of Satze an Sich (sentences in themselves) as the abstract meanings of sentences in his 1837 *Wissenschaftslehre*.[158]

Jacquette raises the possible influence from Bolzano on Frege. *Satze an Sich* is much what Frege has in mind when he takes sentences as the fundamental building blocks of logic and does not ascribe meaning to individual words. He writes in *The Foundations of Arithmetic* three principles:[159]

> First- The psychological is to be sharply separated from the logical, the subjective from the objective;
> Second- The meaning of a word must be inquired after in propositional context, not in isolation;
> Third- The distinction between concept and object is to be kept in sight.

The question may arise: Why not take the second principle too far? If the word cannot be taken in isolation, what makes Frege believe that the proposition can be taken in isolation? The answer

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[157] Ibid, 313.
[158] Gottlob Frege, *The Foundations of Arithmetic* (New York: Pearson Longman, 2007), viii.
[159] Ibid, 17.

to this impacts analytic philosophy to this day. That is, propositions are atomic. The proposition, or sentence, is not merely a grammatical part, like a paragraph. Rather it is the grammatical animal that inhabits logic. The proposition holds a preferred place in logic.

Vassalo argues that even if we take psychologism in Boole to be ratiocination, that is normative logic, psychologism still applies to Boole and Frege equally. She points out that the subtitle of the *Begriffschrift* was “a formal language, modeled upon that of arithmetic, for pure thought.” She describes three positions on what logic is about. First is the view that logic describes thought. This is a view that she says neither man held. Second is the view that logic is the prescription for thought. This is the view that both men were likely to have. Finally is the view that logic just has nothing to do with thought. The operative word in the sub-title of the *Begriffschrift* is “pure.” “Pure” thought is what is common to thought. Remember that Frege said all thought was essentially the same, its concepts are what appear to it. The different views of logic are not as stark as the three positions suggest. Rather Frege was, if nothing else, more vocally anti-psychologistic than Boole. But I also believe that even if Boole didn’t recognize it, he was beginning an anti-psychologistic turn in logic. Both Boole and Frege were leaving behind the more linguistic and psychological logic of previous decades. This puts them somewhere between Vassalo’s second and third categories with Frege closer to the third and far more cognizant that there were categories.

For Frege, the *Begriffschrift* was a tool for finding the foundations of arithmetic. Of the importance of arithmetic considerations to his study of logic, Frege writes: “arithmetic was the point of departure for the train of thought that led me to my ideography.”[160] What was needed after the *Begriffschrift* was a philosophical treatment of the relationship between logic and

[160] Frege, “Begriffschrift”, 8.

arithmetic. *The Foundations of Arithmetic* was published in 1884 and outlines Frege's understanding of this relationship. Frege shared this interest with Richard Dedekind and Giuseppe Peano. Frege criticizes previous logicians like Boole when he writes: "These deviations from what is traditional find their justification in the fact that logic has hitherto always followed ordinary language and grammar too closely."[161] In this passage Frege tells us that his ideography may seem strange, but only because he has thrown off the yoke of grammar. Frege also vehemently attacks the psychological tradition in logic. Jacquette writes:

> He[Frege] rejects and polemicizes vehemently against psychologism in logic and semantics, and his objections have inspired generations of extensionalistically-minded logicians and meaning theorists also to avoid psychological, phenomenological, or generally intentional factors in understanding the nature of meaning.[162]

The traditional historiography is that Frege marks the first instance of strong anti-psychological feelings in logic. However, as we have seen, Frege's view of logic is complicated and perhaps not all that different from Boole's.

Frege provides a criticism of psychologism in logic, but also a deep criticism of Boole's program. He writes:

> Thought is in essentials everywhere the same: there is no question as to whether for different objects there are different kinds of laws of thought. The differences consist only in the greater or less purity and independence of psychological influences, and on external aids to thought, such as language, number signs, and the like, than, say, on the fineness of the structures of concepts;[163]

Frege accepted "different kinds of laws of thought." But he said that there was only one fundamental logic. Boole's calculus was just another calculus, it was not fundamental. To chase a logical notation is only to change the external aid. Boole would agree to this as when he believed

[161] Ibid, 7.
[162] Frege, *Foundations*, ix.
[163] Ibid, 12.

that his algebra was a kind of mathematics, not a foundation for mathematics. But Frege went further. Boole must have thought he was doing something fundamentally novel when he developed an algebra of classes. That thought is everywhere the same is the impetus for removing psychology, and taking up the standard of logicism. Thought in its essentials is everywhere the same because it is fundamentally the recognition of concepts external to it. Our concepts are the logical way of dividing up the world, not quirks of language or thought. After all, what Frege developed was a *Begriffschrift*, a concept writing.

The Foundations of Arithmetic

Frege's second great work was *The Foundations of Arithmetic*. Here he develops logicism, which was so influential to the remaining figures this thesis will discuss. Frege engages with previous mathematicians in *The Foundations of Arithmetic* and we will discuss two of these engagements. First, Frege engages with the eighteenth century German philosopher Immanuel Kant. Kant's position is known as intuitionism. In the chapter on Cantor, we will see Leopold Kronecker hold a similar view. Kant's views will be covered later. Frege criticizes Kant's position when he says:

> Do we have an intuition of 135664 fingers or points at all? If we had, and if we had one of 37863 fingers and one of 173527 fingers, then the correctness of our equation (135664 + 37863 = 173527), if it were unprovable, would need to be immediately obvious, at least for fingers; but this is not the case.[164]

Frege argues that an equation like 135664 + 37863 = 173527, cannot be known by any intuition. It can only be known by a proof. We cannot picture in our minds what 135664 fingers look like. Kant may argue that intuition applies to small numbers only. Frege points out that the distinction between small and large numbers is arbitrary. Frege discusses the definition of number given by

[164] Ibid, 21.

John Stuart Mill, who we saw in the first chapter. Mill wants to say that our concept of number is at heart empirical. Frege writes of Mill's view: "How excellent, indeed, then, that not everything in the world is riveted and nailed down; for then we could not undertake this separation, and 2+1 would not be 3!"[165] For Mill the difference between four and five is the difference between four objects in the world and five objects in the world and just that. Frege derisively called this "gingerbread or pebble arithmetic"[166]. Frege points out that what we do with numbers is hold them in our mind regardless of what the circumstances are in the external world. Mill supposes number to be an adjective like "red" or "horse-shoe shaped." This is the view Frege attacks. He points out that it is wrong to talk about a number applying differently as things are gathered in one spot or dispersed. Frege writes: "Are a thousand grains of wheat, when once they have been sown, no longer a thousand grains of wheat?"[167] He also writes: "One pair of boots can be the same visible and tangible appearance as two boots."[168] Frege believes that number must be separated from the world. We must be able to count thoughts and eggs. So number cannot simply be an adjective, as Mill believes. His view can be better understood in the following way. In recent years popular physics books have been written which postulate the existence of fundamental physical values in the universe such that if they were different this universe would be a very different place, and perhaps not habitable.[169] For Mill numbers are founded in such a way. 2 + 2 = 4 is true in this universe. If it were not, the universe would be very different. Frege's attack on Mill probably stemmed from the fact that its empirical flavor meant that logicism

[165] Ibid, 23.
[166] Ibid, 15.
[167] Ibid, 37.
[168] Ibid, 39.
[169] See Martin Reeses *Just Six Numbers*.

would get us nowhere in understanding number. Logicism, however far it might get us, would become a dead end in the world.

In *The Foundations of Arithmetic* Frege picked up on one-to-one correspondence. This concept leaves the set theory strain where it began with Bolzano and enters the logic strain. Frege would use one-to-one correspondence in a logical and philosophical context. Frege uses one-to-one correspondence in the definition of equinumerosity. Two concepts are equinumerous if a mutual univocal correspondence (MUC) can be drawn between them. This idea needs to be fleshed out a little. The concept of two socks is different from the concept of one pair of socks. The difference is by definition and so logical. The concept of five oranges is not equinumerous with the concept of ten socks. However, if we have the concept of five pairs of socks then there is a mutual univocal correlation between socks and oranges, so then they are equinumerous. This is in contrast with Mill's view. For Frege, a heap of ten pairs of socks is different from a heap of twenty socks; for Mill they are the same. MUC divides concepts into groups that Frege asserts defines numbers. The number two is the set of all couples. The requirement for a concept's admission into this set is MUC with a couple. But Frege does not have the numbers yet. He must first, starting with zero, give us numbers for which we can say all concepts they belong to have MUC. The first number Frege builds is 0. He builds this by pointing out that "0" belongs to the concept "identical to '0', and not identical to '0'" for which no number falls under it. Frege writes:

> It must now be permissible to prove, by means of the previous suppositions, that every concept under which nothing falls, and only with such a one, from which it follows that 0 is the number that belongs to such a concept, and that no object falls under a concept if the number that belongs to this is 0.[170]

[170] Frege, *Foundations*, 77.

Frege uses this sort of argument to construct the successor of 0.[171] A very similar method is employed by Frege in his discussion of infinity. He writes:

> The finite stand opposed to the infinite numbers. The number that belongs to the concept 'finite number' is an infinite one. We designate it, say, by ∞^1! If it were finite, then it could not follow after itself in the natural number series. One can, however, show that ∞^1 does that. In the so defined infinite number ∞^1 there occurs nothing that is in any way mysterious or marvelous. 'The number that belongs to the concept F is ∞^1' now means nothing more and nothing less than: there exists a relation that mutually univocally [one-to-one] correlates the objects falling under the concept F with the finite numbers.[172]

In the above quote we see again the power of the method which Frege applied in constructing "0." Frege gets from the finite to the infinite by saying that the finite numbers fall under the concept "finite number", the number that belongs to this concept is an infinite number. Frege's influence was great in the work of Richard Dedekind and Georg Cantor, who will be discussed in later chapters. On display here is the power of Frege's style of argument. The distinction between "falling under" and "belonging to" along with MUC allows Frege to construct not only finite numbers but also infinity.

The logicism that Frege helped establish can be seen as a reconceptualization of Immanuel Kant's synthetic *a priori* into the analytic *a priori*. This means removing psychology from mathematics. The synthetic *a priori* refers to truths that are known before any particular sense data, but are synthesized out of whole cloth in the mind. The analytic *a priori* are truths that are known before any sense data but are true by logical definition; for example: All mothers are women. Kant placed mathematics in the synthetic *a priori*. This meant injecting psychology into mathematics. Kant did this by dividing all of math into two mental intuitions, the intuition of space, and the intuition of time. Space gives rise to geometry, and time gives rise to arithmetic.

[171] We must make a distinction between a number that "falls under", and a number that "belongs to" a concept. The number that falls under the concept "identical to 0" is 0. The number that belongs to the concept "identical to 0" is 1. A full explication of this part of Frege's argument requires too much room here for explication. See appendix.
[172] Frege, *Foundations*, 83.

Moving mathematics into the analytic *a priori* means that Frege will make the propositions of math akin to: all mothers are women. That is, a proposition of mathematics will be true or false depending on its congruence with the laws of logic. Doing this we now have to be able to analyze mathematical propositions and this means logic. The historian Jose Ferreiros writes:

> The traditional conception of axioms regarded them as true propositions that do not admit of a proof. To the influential Kantian epistemology, axioms were "synthetic a priori principles, insofar as they are immediately true" (Kant 1787, 760). Since logic was a purely analytical science, in which no synthetic principle plays a role, the use of axioms was radically foreign to it. This Kantian conception can still be found in the work of the German mathematical logicians Schroder and Frege. It was for that reason that Frege did not talk about arithmetical "axioms", but about the "fundamental laws" of arithmetic (Frege 1893/1903). Similarly, Dedekind seems to have thought that the logicist program demanded the development of arithmetic in a rigorously deductive way, without any recourse to axioms.[173]

It is important to note that "axiom" is not to be imagined in the formalist sense. Axioms are not chosen they are discovered. They are "true propositions that do not admit of a proof" as Leibniz says. This point will be even more important for Giuseppe Peano. Ferreiros is correct here. Despite his attack, Frege was living in a world Kant created. Although Frege was housing mathematics in different neighborhood he was still housing it in a Kantian town.

The historian Massimo Mugnai characterizes Frege's concerns with logicism when he writes: "Frege . . . considers mathematics as based on logic and constructs a new logical theory to express mathematical truths and to guarantee a perfect control of mathematical proofs."[174] Frege made logic the foundation for mathematics. The system in the *Begriffschrift* was this foundation; it was to be a language for science and mathematics because its subject matter was the same. This would work because logic was about concepts external to the mind similarly to

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[173] Jose Ferreiros, "Traditional Logic and the Early History of Sets, 1854-1908," *Archive for History of Exact Sciences* 50 (1996): 49.
[174] Mugnai, "Logic and Mathematics", 311.

the way that science was. Frege was a powerful ally to set theory because in his philosophy the set is what is fundamental to number. Set theory needed a fundamental account of the natural numbers in order to grapple with the natural number series. Frege, Dedekind, and Peano were forerunners to logicism. The historian of mathematics Howard Eves writes:

> We have seen how these foundations were established in the real number system and then how they were pushed back from the real number system to the natural number system, and thence into set theory. Since the theory of classes is an essential part of logic, the idea of reducing mathematics to logic certainly suggests itself.[175]

Here the algebra of classes we saw with Boole, and will see with Peano, leads to a natural analog with sets. Mathematical logic lent itself to being a foundation for mathematics. It is not an accident that Frege would not only compose a foundation for arithmetic that would have a great influence on set theory, but he also wrote one of the most influential systems of mathematical logic.

Conclusion

The *Begriffschrift* had a great influence on both the set theory and the logic strains. Frege developed a notation in the *Begriffschrift* for sequences and the order of its members. But what this notation accomplished was that it laid a conceptual foundation for speaking about the formation of a sequence and its order. His concept of inheritance in a sequence, as we will see, was mimicked by Richard Dedekind. Frege also had an atomized negation that allowed it to be applied to or stripped from an argument with no damage to the argument. Today the *Begriffschrift* is considered a foundational text in mathematical logic; this chapter has also taken this view. Frege was also a founding logicist. *The Foundations of Arithmetic* is where Frege stated his logicism most succinctly. In that work Frege builds up arithmetic from concepts. We

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[175] Howard Eves, *An Introduction to the History of Mathematics* (Pacific Grove, CA: Thomson Brooks/Cole, 1990), 629.

saw this in the MUC argument where Frege makes use of the difference between falling under and belonging to a concept. Frege's logic, as he says himself, was the driving engine to search for the foundations of arithmetic. In the following chapters we will see that Frege was widely read by the mathematicians who would found set theory.

5: The Manhood of Logic

> Propositions which are deduced from others by the operations of logic are theorems; those for which this is not true I have called axioms. There are nine axioms here, and they express fundamental properties of the undefined signs.
>
> -Giuseppe Peano[176]

Introduction

The logicist program of Gottlob Frege was, for all its novelty, ensconced in a philosophical tradition. *The Foundations of Arithmetic* was not something that would have found its way on to the bookshelf of most nineteenth century mathematicians. The same was true for the *Begriffschrift*; N.I. Styazkhin writes:

> Unfortunately, Frege's *Begriffschrift* was ignored by both philosophers and mathematicians. The former were afraid of the complicated mathematical apparatus; the latter, of the use of such terminology which they, as specialists, considered "typically metaphysical"[177][178]

While this may be true for the *Begriffschrift*, I will show that Cantor and Dedekind were both influenced by *The Foundations of Arithmetic*. Nonetheless, *The Foundations of Arithmetic* also suffered from a lack of mathematical rigor. It is one thing to state that there is a logical foundation for mathematics, it is another to construct one out of logic, and it is still another to

[176] Giuseppe Peano, "Principles of Arithmetic" In *Selected Works of Giuseppe Peano* (Toronto:University of Toronto Press, 1973), 102.
[177] Styazhkin, *History of Mathematical Logic*, 266.
[178] This seems to be a recurring theme in the history of logic. Logic has no nest to roost. In 1914 a young Norbert Wiener wrote to Bertrand Russell about Gottingen, "As usual, the Mathematicians will have nothing to do with anything so philosophical as logic, while the philosophers will have nothing to do with anything so mathematical as symbols." See: Bertrand Russell, *The Autobiography of Bertrand Russell* vol 2 (Boston: Little, Brown and Company, 1967), 39.

hand the mathematician a tool. There are few mathematical concepts in the *Begriffschrift*. It was Frege's program to get at the logic beyond mathematics per se. I will argue that Peano fits within the logicist tradition despite the influence he would have on David Hilbert and Formalism. This chapter will set out the accomplishments of Peano; I will argue that despite the linguistic barrier Peano fits within the research environment which consisted of Frege and Richard Dedekind.

Peano's Philosophy of Mathematics

As we saw in the last chapter, foundational mathematics was coming under the strong influence of philosophy. Frege had founded logicism, particularly through his book *The Foundations of Arithmetic.* Giuseppe Peano was another early founder of logicism. A critique of this view would be that Peano was an early inspiration for formalism. But in the nineteenth century it is not particularly helpful to dwell on this, as formalism had to wait for David Hilbert to become anything like a school of philosophy. Historically we cannot tell much difference between proto-formalism and logicism. Formalism was a school that did not split away from logicism until after the period we are discussing. Hilbert himself was influenced by the logicist tome Principia Mathematica, as Wilder writes:

> Partly influenced, no doubt, by the work of Peano and his school as well as by the Russell-Whitehead work (there is evidence that for a time Hilbert was greatly impressed by the thesis that mathematics can be derived from primitive notions of logic), Hilbert decided upon a union of the axiomatic and logistic methods.[179]

Peano's approach was certainly more axiomatic than any of his day, but he still fell within logicism. I think this idea of formalism rising from a later split is convincing. It is convincing because Hilbert himself states that Russell and Whitehead's logicism was influential to him. Also, Russell himself talks about the great influence Peano had upon him early in his career.

[179] Wilder, *Introduction*, 250.

Russell writes about meeting Peano for the first time at the International Congress of Philosophy in Paris in 1900:

> In discussions at the Congress I observed that he was always more precise than anyone else, and that he invariably got the better of any argument upon which he embarked. As the days went by, I decided that this must be owing to his mathematical logic.[180]

Speaking of Peano's notation Russell wrote:

> It became clear to me that his notation afforded an instrument of logical analysis such as I had been seeking for years, and that by studying him I was acquiring a new and powerful technique for the work that I had long wanted to do.[181]

Peano's system of notation had great effect upon the generation of logicians after him. This notation allowed the mathematician to rigorously state the foundation of mathematics. The first of Peano's papers we will discuss is a pamphlet entitled *The Principles of Arithmetic* (1889). In this work the algebra of classes mixed with logicism to create a foundation. In the work of Peano we see the logic of Frege and Boole mixed together. He was next only to Frege as the great advocate of logicism in the nineteenth century.

Peano's System

As mentioned in the previous chapter Frege, working in the Kantian tradition, did not want to explicitly refer to axioms lest mathematics should fall into the synthetic *a priori*. This was a concern Peano didn't have and he was quite comfortable with the idea of axioms of arithmetic. Peano developed nine axioms that were to form the foundation on which arithmetic was to be built. These are stated as follows, though not in their symbolic form:[182]

1. One is a natural number.

2. If a is a natural number then $a=a$.

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[180] Bertrand Russell, *The Autobiography of Bertrand Russell* vol 1 (Boston: Little, Brown and Company, 1967), 218.
[181] Ibid.
[182] Translations by author. Peano, "Principles", 113.

3. If *a* and *b* are natural numbers then $a=b$ is equivalent to $b=a$.[183]

4. If *a*, *b*, and *c* are natural numbers then if $a=b$ and $b=c$ then $a=c$.

5. If $a=b$ and *b* is a natural number then *a* is a natural number.

6. If *a* is a natural number then $a+1$ is a natural number.

7. If *a* and *b* are natural numbers and $a=b$, then $a+1 = b+1$.

8. If *a* is a natural number then $a+1$ cannot equal 1.

9. If *k* is a class and 1 is a member of *k* and *x* is a member of *k* and *x* is a natural number, then if $x+1$ is a member of the class *k* then all the natural numbers are members of the class *k*.

These axioms set the definitions of natural numbers and how they were to behave. Three of Peano's axioms are particularly important for us. The first axiom states that 1 is a natural number. It is important to note here that this axiom does not state that 1 is the first natural number. The eighth axiom excludes 0 from the natural numbers. This is because if 0 is a natural number then 0+1=1, which violates the eighth axiom. The ninth axiom has come to be known as the mathematical induction principle and will be important in our discussion of Richard Dedekind. This axiom allows mathematicians to prove statements about natural numbers.

Peano's work was part of a new movement in mathematical research into the foundations of arithmetic. This movement looked to logic as a way of constructing mathematics. Peano formed a research environment along with Gottlob Frege and Richard Dedekind. It was this environment that would eventually overflow into Georg Cantor's set theory. Styazkhin writes:

> It should be noted that Peano's axiomatization was to a significant degree inspired by the ideas expressed by R. Dedekind in his treatise. In general, Peano was under the strong influence of the theory of functions developed by Dedekind.[184]

[183] "equivalent to" here is written as "=", I have decided to translate the symbols this way because in the cases where "=" is
the main operator the meaning is not to be taken as an equivalence of quantity; rather it is to be taken as bi-conditional, it could then be translated as "if and only if."

Peano took up the logicism of Frege, although he would also be influential in formalism.[185] However, Peano cites Boole as having a great influence on *The Principles of Arithmetic* (1891) and in places he uses Boole's notation. *Principles of Arithmetic* contains the law of idempotence, $aa=a$, just as *The Mathematical Analysis of Logic* does. Peano was not wary of the psychologism of Boole's system, perhaps because of the use of arithmetic in his logic. It might have been that logic didn't seem psychological to Peano. Peano agrees with Boole's quantification of the predicate. $A = B$ for Peano can be symbolized as $(A > B) \wedge (A < B)$. In his notation he melds the algebra of classes with the propositional logic of Frege and the logicist program N.I. Styazkin writes:

> Rather, the significance of Peano's total achievement lies in its being a transitional link between the algebra of logic (in the form given by Boole, Schroder, Peirce, and Poretskiy) and the contemporary form of mathematical logic.[186]

By "contemporary form" Styazkhin means propositional logic, the logic of Frege. Peano resolves the divide between Frege's and Boole's notation.

The "Ɔ" used by Peano symbolizes a Frege style implication. In *The Principles of Arithmetic* Peano mentions Frege's implication as an analog for his own. In 1890 Peano was familiar with the *Begriffschrift*. Peano also symbolizes class membership, in Boole's sense, with "ε." Here we can see the melding of the algebra of classes and propositional logic. It is useful, now, to go into an explication of Peano's notation in *The Principles of Arithmetic*. As mentioned above, Peano splits the class and propositional interpretation of logic. He allows for class membership and implication. Peano introduces the signs ., :, ∴, and ::. These replace the use of

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[184] Styazhkin, *History of Mathematical Logic*, 279.
[185] It is important to mention here that Peano mentions axioms. As mentioned in the chapter on Frege, axiom is not to be taken in a formalist sense.
[186] Styazhkin, *History of Mathematical Logic*, 276.

parentheses. $(((ab)(cd))((ef)(gh)))k$ would be symbolized as, $ab.cd{:}ef.gh \therefore k$.[187] Peano uses a backwards "ε," this can be translated as "such that." On the left side is a variable, or variables, on the right side are the conditions they must satisfy. This would be used to signify a proposition like "an X such that it is a natural number and is greater than all natural numbers does not exist." "Such that" will sometimes be symbolized as "[ε]." Peano calls this "inversion." He symbolizes the natural numbers as N, the rational numbers as R, and the real numbers as Q. The positive real numbers he calls quantities. "K" symbolizes a class, it can be taken as the class of classes. So, aεKN means that *a* is a class of natural numbers. "Λ" is the notation for the empty class. Peano calls "Λ" absurdity. He writes:

> Propositions (I)(A) and (II)(E) cannot coexist, supposing that class A is not empty. Certainly, when the logicians affirm that two contrary propositions cannot coexist, they understand that class A is not empty; but although all the rules given by the preceding formulas are true no matter what the classes which make them up, including 0 and 1, this is the first case in which it is necessary to suppose that one of the classes considered is not empty.[188]

We have mentioned in earlier chapters that logicians had long disallowed the existence of empty classes in their logics. What Peano is telling us here is that making A and E mutually exclusive is the first place in his system where one must follow suit with the earlier logicians. What is important here is that Peano is treating the quantification of the predicate as an exceptional case. Peano disallows the the empty class only to fit one interpretation. Another interpretation, using the notation in the quote above, is to say that if A and E can exist then *A* is empty. The choice is left to the logician.

Peano introduces the symbol $\supset_x$. He uses it in symbolizing his ninth axiom. Peano makes two statements of interest to us on this topic in his 1897 article "Studies in Mathematical Logic."

[187] Peano, "Principles", 104.
[188] Giuseppe Peano, "The Geometrical Calculus" In *Selected Works of Giuseppe Peano* (Toronto:University of Toronto Press, 1973), 87.

He writes that p $\supset_{x,\ldots,z}$ q means, "'whatever x,…,z are, as long as they satisfy the condition *p*, they will satisfy the condition *q*.' The indices to the sign Ɔ are omitted when there is no danger of ambiguity."[189] This notation is used if variables like *x* and *y* are used in *p* or *q*. The notation is necessary in Peano's ninth axiom because the hypothesis of the implication contains the variables *k* and *x*. The notation holds continuity from the value of *x* in the hypothesis to the consequent. *x*+1 must be made to be the successor of *x* in the hypothesis.[190]

Peano was familiar with point sets, and more importantly Cantor's work in them. Peano writes: "Finally, in S10 I have given several theorems, which I believe to be new, pertaining to the theory of those entities which Professor Cantor has called Punktmenge."[191] Punktmenge here is translated as "point set." The section of *Principles of Arithmetic* Peano is referring to is the section on quantities, or real numbers. Georg Cantor, Richard Dedekind, and Peano are dealing with the location of points in a set. Phillip Jourdain writes of Cantor's consideration on this subject:

> If we are given a system (P) of points in a finite interval, and understand by the word "limit- point" a point of the straight line (not necessarily of P) such that in any interval within which this point is contained there is an infinity of points of P, we can prove Weierstrass's theorem that, if P is infinite, it has at least one limit-point. Every point of P which is not a limit-point of P was called by Cantor an "isolated" point.[192]

Discussing Peano's consideration of this subject will show what was of concern with limit points. Peano creates three symbols: Ia, Ea, and La. These are respectively, interior point, exterior point, and limit point. Peano's interior point is Cantor's isolated point. Let us now look

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[189] Giuseppe Peano, "Studies in Mathematical Logic" In Selected Works of Giuseppe Peano (Toronto:University of Toronto press, 1973), 193.
[190] The ninth axiom in symbols is: $k \varepsilon K \therefore 1 \varepsilon N \therefore x \varepsilon N : \supset_x . x{+}1 \varepsilon k :: \supset . N \supset k$
[191] Peano, "Principles", 102.
[192] Georg Cantor, *Contributions to the Founding of the Theory of Transfinite Numbers* (La Salle, Illinois: Open Court), 30.

at what Peano was saying about these points. Here are two translated definitions Peano gives us in *Principles of Arithmetic*[193],

If a is a class of quantities then, Ea = I(-a)

If a is a class of quantities then, La = (-Ia)(-Ea)

The "-" sign denotes negation. The first proposition states that an exterior point to class *a* is an interior point to class -*a*. The second proposition states that a limit point is neither an interior point nor an exterior point. Limit points were a point of overlap between logic and set theory because logic delved into the construction of the use of points in Peano. These points were discussed heavily in the formative years of set theory. They became important enough to mathematicians that Peano felt the need to construct them.

Peano's investigations were repeated by other figures in this narrative. His twenty-third theorem under the section "Subtraction" was repeated by both Frege and Dedekind. It states that, "If a and b are natural numbers then, either a < b, or a = b, or a > b."[194] This is a proposition about rank and it is the most fundamental proposition about rank. Peano also delved into the order of the members of a class. Cantor and Dedekind had similar treatments. He provides three definitions concerning what he calls the maxima and minima. The maxima is the greatest member of a class, and minima is the least member of a class. Peano writes:

> If *a* is a class of natural numbers then the maxima of *a* is a number *x* such that *x* is a member of *a* and a member of *a* such that it is greater than *x* does not exist.
>
> If *a* is a class of natural numbers then the minima of *a* is a number *x* such that *x* is a member of *a* and a member of *a* such that it is less than *x* does not exist.
>
> If *n* is a natural number and *a* is a class of natural numbers and *a* is not empty, and, if there is no member of *a* greater than *n* then, the maxima of *a* is a natural number.[195]

[193] "Definition" here has a special meaning. A definition is a proposition that delimits the use of a symbol. In his 1897 article Peano tells the reader to interpret these definitions as "we name". See Peano, "Studies", 196.
[194] Translated by author. Peano, "Principles", 118. Translation by author.
[195]Translated by author. Peano, "Principles"119. Translation by author.

The concepts of maxima and minima will be important in the work of Dedekind and Cantor. One of the big moves in logic was to become more interested in the members of a class and not just the class as a whole. These figures were more interested in discussing the rank of members of a class.

Peano mentions early in *Principles of Arithmetic* that he especially made use of Boole's algebra of classes. However, the influence of Boole on Peano is perhaps better seen in Peano's 1888 treatise *The Geometrical Calculus*. The opening chapter on deductive logic pertains to an algebra of classes much like Boole's. Peano writes:

> By the expression A ∧ B ∧ C ∧ . . ., or ABC . . ., we mean the largest class contained in the classes A, B, C, or the class formed by all the entities which are at the same time in A and B and C, etc. The sign ∧ is logical conjunction. We shall also call it logical multiplication, and say that the classes A, B, . . . are factors of the product AB. . .[196]

Peano shares the algebraic analogy with Boole. Both men have logical multiplication in their systems. Both are using a class interpretation of logic. Peano, specifically, is doing this after the publication of the *Begriffschrift*. Peano also uses implication as the *Begriffschrift* used it, in *Principles of Arithmetic*.

Peano, like Frege, was also writing a language for science and logic. Peano writes: "I believe, however, that with only these signs of logic the propositions of any science can be expressed."[197] This was perhaps not so big a motivation for Peano as it was for Frege. Peano's attempt seems more directed to the mathematician as his use of axioms allow for building up the language of science and mathematics. Frege's *Begriffschrift* never seemed as concerned with mathematics as *The Principles of Arithmetic* does. Peano stated his system as:

[196] Peano, "Geometrical", 76.
[197] Peano, "Principles", 103.

> One of the most notable results is that, with a very limited number (7) of signs, it is possible to express all imaginable logical relations, so that with the addition of signs to represent the entities of algebra, or geometry, it is possible to express all the propositions of these sciences.[198]

Peano's aim was at a language for mathematics, Frege's was at a language for science. Although as the above shows, even Peano could not help imagine his system working fall all of science. Peano writes about Frege's reaction to his system. He writes: "he (Frege) expresses doubt that my ideography can serve to do more than express propositions."[199] Frege's dream was to develop a logical language with which new knowledge could be created by the combination of symbols. Frege was criticizing Peano's system because he felt that it did not get to the concepts beyond the propositions.

Peano used inclusive disjunction. This can be seen in the example he gives in *The Geometrical Calculus* in which the numbers that are greater than 1 or less than 2 comprise the universe. This would not be the case if disjunction were exclusive because the numbers between 1 and 2 would be left out. Peano states that (A=O) ∧ (B=O)= (A∨B=O). Stating that each individual of a series of classes is empty is logically equivalent to saying that the logical addition of all those classes is an empty class. Peano symbolizes Barbara[200] as: (A < B) ∧ (B < C) < (A < C). In his notation then we would have the proposition A as: A < B. E would be: A < ~B. I would be: ~(A < ~B) . And O would be: ~(A < B). In order to show the sort of proofs Peano was trying to construct we will look at one. Peano puts forth the theorem that 2 is a natural number. He begins by referring to axiom one that 1 is a natural number. He then refers to axiom 6 replacing *a*

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[198] Ibid, 154.
[199] Peano, "Studies", 192.
[200] Barbara is the device taught to logic students to remember the syllogisms. To interpret these you take the vowels out and they will correspond to corners of the logic square. So Barbara would be aaa, or "all A are B, all B are C, therefore all A are C". This is what is symbolized in the following proposition where "<" represents class inclusion, "∧" means conjunction.
[200] Peano, "Geometrical", 88. Peano also symbolizes Barbara in using O as the empty class. I will leave it to the reader to see how this is done.

with 1. He then derives the proposition that the sum of 1 and 1 is also a natural number; he does this with modus ponens. He then makes reference to the definition that 2 is equivalent to 1 + 1. That is, 2 is the symbol for 1 + 1. He finally refers to axiom five. Since 1 + 1 is a natural number and 2 is equivalent to it, then by axiom five 2 is a natural number. One can get a hint of why formalists like Hilbert would look to Peano as an inspiration.

In his 1897 “Studies in Mathematical Logic” Peano introduces the existential quantifier. He translates this as “there exists.” It should be noted that Frege was able to get by with just the universal quantifier. Likewise, because of the same equivalency Peano is able get by with just the existential quantifier. In this same paper Peano also touches upon set theory. He gives the following three theorems:

a, *b* ε K.Ɔ(f ε bfa.=:x ε a.Ɔx.fxεb

a, b ε K. Ɔ(f ε (bfa)Sim.=: f ε bfa: xy ε a. x ~= y. Ɔxy.fx ~= fy

a, b ε K. Ɔ(f ε (bfa)rcp.=: f ε (bfa)Sim : y ε b . Ɔy . ∃ [xε](x ε a.fx = y)□

The first of these theorems states that: If *a* and *b* are classes then; *f* being a correspondence between *b* and *a* is equivalent to, if *x* is a member of *a* then whatever *x* is the *f*-correspondent to *x* is a member of *b*. The second theorem states that: If *a* and *b* are classes then; *f* being a similar correspondence between *b* and *a* is equivalent to, if *f* is a correspondence between *b* and *a*, and *x* and *y* are members of *a*, and *x* does not equal *y* then, *f*-correspondent *x* is not equal to *f*-correspondent *y*. The third theorem states that: if *a* and *b* are classes then; *f* being a reciprocal correspondence between *b* and *a* is equivalent to; if *f* is a similar correspondence between *b* and *a*, and *y* is a member of *b* then, whatever *y* is, there exists an *x* such that *x* is a member of *a* and

[201] Peano, “Studies in Mathematical logic”, 204-205. Here I use square brackets for inversion. The original quote contains a line over “xε”.

the *f*-correspondent of *x* equals *y*. Here we see some of the same observations that we will see with Dedekind.[202]

Conclusion

Peano continued the logicist program from Frege. Formalism was not a recognizable school yet. Frege's comments and Peano's treatment perhaps illustrates the points where formalism began to break away from logicism. As mentioned earlier, Frege's critique of Peano's notation was just that it didn't get to the underlying logic. He made it more mathematically rigorous. With his axioms Peano hoped to create a system in which the proposition of science could be expressed. A fundamental investigation into arithmetic was necessary in the case of Peano and Frege, because of the strange findings that were coming out of the work of Dedekind and Cantor.

[202] See appendix for further explication of Peano's views on correspondences.

6: A Dictionary for Set Theory

Introduction

Richard Dedekind’s investigation of the infinite was very much in line with Bernard Bolzano’s work. Richard Dedekind (1831-1916) was Bolzano’s successor in the investigation of infinity. He meshed infinity with the logicism of Gottlob Frege. Both Dedekind and Frege believed that mathematics rested upon logic.Like Bolzano, Dedekind offered a definition of infinity. The main accomplishments of Dedekind are that he rigorously constructed the real numbers using the Dedekind cut. He also combined logicism with set theory. This system consisted of a vocabulary and elementary theorems derived from the vocabulary. The work of Dedekind is the first time in our narrative that logicism made an impact on considerations of infinity. The foundational program seen in Giuseppe Peano and Frege ran into a concern over infinity in the work of Dedekind. This concern was cross-pollination between the mathematical logic strain and the set theory strain. Dedekind’s investigation was in large part duplicated by Peano. This was another point of contact between the logical strain and the set theory strain. Much of this chapter will be about cross-pollination between Dedekind and Peano on the one hand, and Dedekind and Frege on the other. This chapter will show how Dedekind created a vocabulary for concepts in set theory. This foundational program required a new vocabulary and much of what Dedekind offers us is definitions. The new vocabulary rigorous and necessary for this program was partly created by Dedekind. The works that will be discussed here are “Continuity and Irrational Numbers” (1872) and “The Nature and Meaning of Numbers” (1888).

His program was slightly different from that of Cantor, who will be discussed in the next chapter. “The Nature and Meaning of Numbers” was published in 1888, fourteen years after Cantor published his first paper on set theory. But Cantor’s program was not foundational; Cantor mentions little of constructing arithmetic from sets. Dedekind did try to construct arithmetic from set theory.

Dedekind was born in 1831 in north Germany. He studied mathematics at Göttingen where he studied under Peter Gustav Lejeune Dirichlet and Bernhard Riemann. Both men would have great influence on Dedekind. Dedekind taught at Göttingen and Zurich before ending up at Brunswick, the city where he was born. He taught at a university in Brunswick until his retirement in 1896. Dedekind was part of the wave of German influence that swept over the mathematical world in the nineteenth century. Dedekind studied at Göttingen and for a short period at Berlin. He was involved with the great German mathematicians of his day and in his later years would be an established supporter of Georg Cantor and set theory. This is important, as we will see, because of the criticism Cantor faced when he made his ideas known. Before discussing the body of Dedekind’s work, we should pause for a discussion of Dedekind’s method, particularly in “The Nature and Meaning of Numbers.” These are definition, theorem, and proof. The definitions Dedekind provided gave a rigorous vocabulary: They carry Bolzano’s founding program further. After a new piece of vocabulary is provided, Dedekind immediately sets to stating theorems using the new terms. These theorems are, in turn, proven. The theorems and proofs serve to fill out this new vocabulary and direct Dedekind’s program. How this vocabulary played in a proof added further definition.

Dedekind's Definitions

The first definition we will discuss is Dedekind's definition of infinity. It was he who defined infinity in a rigorous way to allow set theory to be developed. For Dedekind an infinite set was one which could be put into a one-to-one correspondence with a proper subset[203] of itself. For example, the natural numbers can be put into a one-to-one correspondence with the even natural numbers as follows: (1,2)(2,4)(3,6)(4,8)(5,10)(6,12) . . . This makes the set of natural numbers infinite, or Dedekind infinite. Here, the set of natural numbers is placed into a one-to-one correspondence with the even natural numbers by applying the function 2n to the natural numbers. [204][205] On the other hand, the natural numbers one through fifteen cannot be placed into a one-to-one correspondence with the natural numbers from ten to fifteen as is shown: (1,10)(2,11)(3,12)(4,13)(5,14)(6,15)(7,?). The numbers one through fifteen cannot be put into one-to-one correspondence with this proper subset of itself. And, indeed, the set of natural numbers one through fifteen is not infinite. A proper subset in this context has a very precise meaning that is to be distinguished from the term "subset." A subset is any part of a set including the entire set itself. A proper subset is a part of a set, but it is not the entire set itself. The term "proper subset" is comparable with "part" in Bolzano's definition of infinity. The distinction between "subset" and "proper subset" makes Dedekind's definition more rigorous than the holomerism of Bolzano.

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[203] "Proper subset" is used here as opposed to "subset". "Proper Subset" has the precise meaning of being only a part of the set of which it is a "proper subset". "Subset" on the other hand can refer to part of or the whole set of which it is a "subset"

[204] Dedekind uses the term "transform."

[205] A short word on this function. Dedekind calls this a "transformation" and writes: "By a transformation [Abbildung] ϕ of a system S we understand a law according to which to every determinate element s of S there belongs a determinate thing which is called the transform of s and denoted by ϕ(s); we say also that ϕ(s) corresponds to the element s, that ϕ(s) results or is produced from s by the transformation ϕ, that s is transformed into ϕ(s) by the transformation ϕ."(Richard Dedekind, *Essays on the Theory of Numbers* (New York: Dover, 1963), 50) : "S" here is the set of natural numbers. A transform ϕ of each member s of S produces the system S' composed of all the transformed members of S. In our usage the system of even numbers would be S'. The transform would be 2n. When defining a transformation Dedekind cites the mathematician Gustav Dirichlet who will be discussed later. A transform is similar to what is called a mapping, or imaging in Cantor's. Dedekind, in his leeter to Keferstein

The next definition to discuss is that of "simply infinite." "Simply infinite" resembles what Cantor termed the aleph null. Dedekind writes: "All simply infinite systems are similar to the number-series N and consequently by (33) also to one another."[206] This is a fact that Cantor will show with his proofs. The set of natural numbers is the benchmark for the simply infinite. Dedekind is stating the same thing Cantor showed by proving equinumerosity between the natural numbers and the algebraic and rational numbers. Dedekind writes: "A system N is said to be simply infinite when there exists a similar transformation ϕ of N in itself such that N appears as chain (44) of an element not contained in ϕ(N)."[207][208] Dedekind writes "there exists" so we only need to show that there is one such transformation. Dedekind is saying here that a system N is simply infinite if it can be transformed into another system N' that is part of N and such that N is a chain which increases by an element not included in N'.[209] To unpack this further, we saw that the natural number system is a chain that increases by an element of one. As we did earlier, we can apply the transformation n+1 on all the natural numbers and generate the successors of all natural numbers which does not include the number one.[210] Now let us try this with the real numbers. Suppose we applied the transformation n+1 to the real numbers. We would start at (.1, .2) and (1.1, 1.2) but then what about (.01,.02) and (.02,.04). Thus, the real numbers are not a chain by the transformation n+1. In this case we cannot get the successors of real numbers. But also, there is no lowest real number so we can never say that one was not a member of the transform of the real numbers. We would have .0000… and n + 1 would be 1.0000…, which is

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[206] Richard Dedekind, *Essays on the Theory of Numbers* (New York: Dover, 1963), 92.
[207] "similar" in this quote will be defined in the next chapter, as it is a concept properly identified with Georg Cantor. It suffices to say here the "similar" means order preserving. That is, whatever order was in the domain of the transform will be found in the range.
[208] Dedekind, *Essays*, 67.
[209] The concept of a "chain" is discussed below.
[210] 1 is the first natural number.

just the same as one.[211] What is at play here is that it is not possible to get to the next real number by any transform of real numbers. We cannot find a first real number or a next real number. Therefore, the real numbers are not simply infinite.

The next definition we will discuss is that of a "chain." This idea would become central to set theory in the twentieth century. Dedekind writes:

> K is called a *chain* [Kette], when K'эK. We remark expressly that this name does not in itself belong to the part K of the system S, but is given only with respect to the particular transformation ф; with reference to another transformation of the system S in itself K can very well not be a chain.[212][213]

For Dedekind, a chain occurs when a set contains its own transform. For example, the transform 2n creates a chain from the natural numbers since the set of even numbers is itself a part of the natural numbers. It is important to note here that a transform makes a certain chain. "Chain" is not a label that can be affixed to a set, but rather a function is chain forming. The natural numbers are not a chain when the transform n/2 is considered. This transform would give us the set {.5, 1, 1.5, 2, 2.5…} which are not all natural numbers.

Dedekind then turns his definitions on the natural numbers. He defines the concept of the simply infinite when he writes:

> If in the consideration of a simply infinite system N set in order by a transformation ф we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order setting transformation ф, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series N*.[214]

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[211] The proof is: 10n-1n= m; m is 9 whether we use 1 or 1.000…Cantor would use a similar method in his proofs.

[212] Dedekind, *Essays*, 56.

[213] In Dedekind's notation э means membership of the left element in the right element. Whereas in Peano's notation ε is the symbol for membership. In Dedekind's notation A is a member of B would be written A э B; in Peano's notation it would be A ε B. See page 46 where Dedekind introduces э.

[214] Dedekind, *Essays*, 68.

For Dedekind, the natural numbers compose a series, set up by a transform. The relations that are integral to ranking are predecessor and successor. The natural numbers are a series such that the element of transform is one. This concept should receive closer scrutiny. Dedekind writes in his letter to the mathematician Hans Keferstein from 1890: “I define the number 1 as the basic number of the number sequence without any ambiguity in articles 71 and 73.”[215] Dedekind writes in article 71: “We call this element, which we shall denote in what follows by the symbol 1, the base-element of N and say the simply infinite system N is set in order by this transformation ϕ.”[216] In the simply infinite system N every number can be reached from every other number successive increase or decrease by one. Peano’s successor axiom similarly has working behind it the base element one. Dedekind recognized the same similarity with Fege in his letter to Keferstein when he writes:

> Frege’s *Begriffschrift* and *Grundlagen der Arithmetic* came into my possession for the first time for a brief period last summer (1889), and I noted with pleasure that his way of defining the non-immediate succession of an element upon another in a sequence agrees in essence with my notion of a chain; only, one must not be put off by his somewhat inconvenient terminology.[217]

To clarify this further if a system K is a chain then a transform ϕ(K) produces a system K’ such that K’ is a subset of K. The system K is simply infinite if there is a transform of an element such that element is not contained in K’. The best example for the system of natural numbers is the transform into the system of successors of natural numbers. For this, we would set up a transform $n + 1$.

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[215] Richard Dedekind, “Letter to Keferstein” in *From Frege to Godel* (Cambridge, Massachusetts: Harvard University Press, 1967), 102.
[216] Dedekind, *Essays*, 67.
[217] Dedekind, “Keferstein”, 100.

Dedekind's Construction of the Real Numbers

Dedekind engages with other writers in this history. This engagement centers on Dedekind's ideas in "The Nature and Meaning of Numbers." Dedekind, Peano, Bolzano, and Frege read each other; however, the engagement goes deeper than this. Many of the results these mathematicians reach were the same. Dedekind cites Bolzano's proof of the existence of the infinite. Dedekind's treatment of infinity is similar to Bolzano's. Ferreiros writes of this:

> It is well known that, in the meantime, Dedekind read Bolzano's *Paradoxien des Unendlichen*, where a similar proof is presented (Bolzano 1851, S13). Bolzano's proof seems to have motivated Dedekind to include this theorem,[218]

Dedekind was also influenced by Georg Cantor who will be discussed in the next chapter. Dedekind was in correspondence with Cantor, although he does not cite Cantor often. Cantor was almost underway in his own considerations of sets when Dedekind wrote these two papers. In "Continuity and Irrational Numbers," Dedekind mentions having received a paper by Cantor on trigonometric series in 1872. Dedekind also cited Leopold Kronecker who as we will see had a vicious debate with Cantor over the latter's transfinite arithmetic.[219] Dedekind felt that Kroenecker's approach to the "logic which deals with the theory of numbers"[220] was unsatisfactory. Dedekind criticizes Kroenecker's intuitionism, a sign that Dedekind had logicistic leanings. As already mentioned, Bolzano defined of infinity. Dedekind stated infinity more rigorously and it is his infinity that has since been adopted by mathematicians. The historian Jose Ferreiros writes of this:

> Bolzano tried to build up a precise theory of the mathematical infinite, but after being close to the right point of view, he departed from it in quite a strange direction. Bolzano stated clearly the fact that two infinite sets can be put in a one-to-one correspondence

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[218] Ferreiros, "Traditional Logic", 55.
[219] Dedekind, *Essays*, 31, 45.
[220] Ibid, 31.

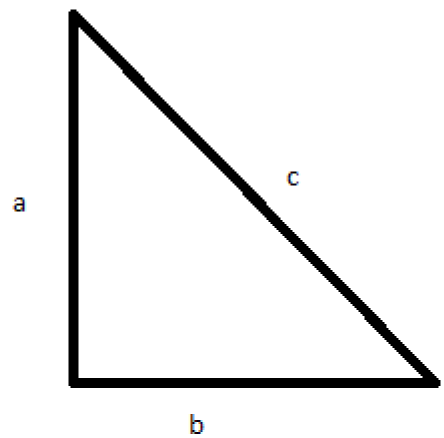
> while one of them is a subset of the other. (Bolzano *1851*, 27-28) But from this he did not conclude that they have equal cardinality.[221]

Dedekind continued the practice of the research environment of describing the arrangement of members of a set. What he also did was expand discussion from arithmetic to numbers more generally. He brought logicism to the real number system. Dedekind was concerned with constructing the real numbers. Constructing the real numbers, by definition, means constructing the rational and irrational numbers. The rational numbers had long been well defined and constructed. The irrational numbers were not defined except negatively as "not rational." Mathematicians acknowledged that irrational numbers exist ever since the ancient Greeks. They could name some irrational numbers, for example $\sqrt{2}$. Dedekind made irrational numbers almost as well understood as rational numbers. He accomplished this with his most well-known idea, the Dedekind Cut.

To discuss where the Dedekind Cut came from, it is necessary to understand the history of irrational numbers.[222] Irrational numbers were discovered by the Pythagorean mathematicians of ancient Greece. It was believed at the time that all the points on a line corresponded to rational numbers. The rational numbers between 0 and 1 can be thought of as an integer over another integer that divides a unity length. So the number 1/5 says that we divide the unit length into five equal parts, 1/5 is one of these parts. Mathematicians had faith that two lines could always be compared with rational numbers. This is because they believed that there was always a way to divide one line into equal units that would in turn divide into the second line a whole number of times, whether they be fifths, sixths, twentieths, etc. What the Pythagoreans discovered was that

[221] Ferreiros, "Traditional Logic", 2.
[222] This account is taken from Eves' *Introduction*.

this comparison is not always possible. The most well-known irrational number is $\sqrt{2}$.[223] If one considers the Pythagorean Theorem for a triangle with sides of unit length one will see the crisis that irrational numbers cause.



The Pythagorean Theorem is written as $a^2 + b^2 = c^2$. If a triangle has sides of unit length, then this becomes $1 + 1 = 2$. Thus, finding the hypotenuse requires calculating the square root of two. What we find is that there is no way to break up the unit length into equal pieces such that one of these pieces will divide c a whole number of times. Hence, the problem with this triangle above is that c being irrational pulls it out of the paradigm of measurement that was used for the sides a and b.

The Dedekind cut was expounded in an 1872 paper titled "Continuity and Irrational Numbers." Dedekind was trying to construct the real number system but as he himself admitted:

> But what advantage will be gained by even a purely abstract definition of real numbers of a higher type, I am as yet unable to see, conceiving as I do of the domain of real numbers as complete in itself.[224]

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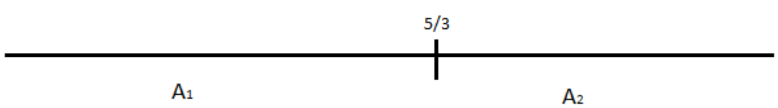
[223] A proof of the irrationality of $\sqrt{2}$ is offered in the appendix.
[224] Dedekind, *Essays*, 3.

It should be noted that Dedekind is providing a new definition of the real numbers. But this is only as good a definition as it is useful. We will go into some of the powerful usefulness of this definition. Dedekind begins by relating arithmetic to geometry through an analogy between real numbers and geometric line. If we lay down a stick √2 units long starting at 0 the tip will fall on a point that corresponds to no rational number. If, like the Greeks, we supposed our ruler to consist only of rational numbers, then we may be surprised to find a gap, a Dedekind gap. Dedekind says that gaps like these are filled with the irrational numbers.

Dedekind defines rational and irrational numbers with a cut. He begins by instructing us to take the system of rational numbers and to put them on a number line. We choose one number and say all numbers less than this number are members of the set A_1, and all numbers greater than this number are members of the set A_2. Take the illustration below:



Here we can see that the cut is performed at the rational number 5/3. Dedekind tells us that the mathematician has the freedom to choose which set (A_1 or A_2) 5/3 itself is to belong to. Furthermore, this cut--and any cut that corresponds to a rational number splits the rational number system into two parts. Now, since we have put 5/3 into A_1 or A_2 either A1 has a greatest member, 5/3, or A2 has a least member, 5/3. This, then, is the definition of a rational number. Dedekind writes: “The property that either among the numbers of the first class there exists a greatest or among the numbers of the second class a least number.”[225] A rational number then is defined by the cut it forms. Next Dedekind provides a proof of the existence of irrational

[225] Ibid, 13.

numbers using the idea of a cut.[226] He shows that there are certain cuts that can be made such that the cut does not meet the definition above of a cut at a rational number. If we make a cut and A_1 has no greatest member and A_2 has no least member, then the cut corresponds to no rational number. As Dedekind writes: "Whenever, then, we have to do a cut (A_1, A_2) produced by no rational number, we create a new, an irrational number *a*, which we regard as completely defined by this cut."[227] I will mention only one example of what can be done with cuts. Dedekind used cuts to define addition between real numbers. The equation $a + b = c$ can be defined in terms of cuts. The numbers *a*, *b*, and *c* are real numbers. We imagine that *a* and *b* describe cuts.[228] Dedekind describes addition thus. We place in C_1 all real numbers *c* such that $a + b \geq c$. In C_2 we place all numbers c' such that $a + b \leq c$. This addition then is a function in which any member of A_1 added to any member of B_1 yields a member of C_1. Any member of A_2 added to any member of B_2 yields a member of C_2.[229] Therefore, $a + b$ describes a cut *c* in the real numbers.[230]

Dedekind's Part in the Research Environment

Dedekind was the figure in the set theoretical strain to have the most interaction with Frege and Peano. Dedekind mentions having read Frege's *The Foundations of Arithmetic*. He writes:

> About a year after the publication of my memoir I became acquainted with G. Frege's *Grundlagen der Arithmetik*, which had already appeared in the year 1884. However, different the view of the essence of number adopted in that work is from my own, yet it contains, particularly from S 79 on, points of very close contact with my paper, especially with my definition (44). The agreement, to be sure, is not easy to discover on account of the different form of expression; but the positiveness with which the author speaks of the logical inference from n to n+1 (page 93, below) shows plainly that here he stands upon the same ground with me.[231]

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[226] An exposition of this proof is located in the appendix.
[227] Dedekind, *Essays*, 15.
[228] See appendix for illustration.
[229] In modern set theory we would say that ($a_1 + b_1$) maps onto C_1, ($a_2 + b_2$) maps onto C_2.
[230] See appendix for one of Dedekind's proofs utilizing cuts.
[231] Dedekind, *Essays*, 42-43.

What Dedekind claims here is that he and Frege have about the same idea as to how succession in a series might be logically constructed. But like Peano, Dedekind offered the reader more mathematical rigor than Frege. He cites the German logician Ernst Schroder. He mentions Schroder's *Lehrbruck der Arithmetik und Algebra,* writing, possibly in response to some criticism Schroder made of his notation:

> I will simply confess that in spite of the remark made on p.253 of Part I., I have retained my somewhat clumsy symbols (8) and (17); they make no claim to be adopted generally but are intended simply to serve the purpose of this arithmetic paper to which in my view they are better adapted than sum and product symbols.[232]

It will be seen later that Frege made a similar criticism.

Dedekind was also paralleling work done by Giuseppe Peano. The Dedekind's work was more concerned with set theory than "Principles" was. Peano would write on set theory later in his career. In the time period this thesis covers there is little mention of sets. Dedekind, on the other hand, conceived of his logicist program from the vantage point of set theory. In the chapter on Peano we saw that Peano's ninth axiom was the axiom of mathematical (or complete) induction and was stated thus:

> If k is a class and 1 is a member of k and x is a member of k and x is a natural number, then x+1 is a member of the class k which in turn means that all the natural numbers are members of the class k.

In "The Nature and Meaning of Numbers," Dedekind relies heavily on complete induction. He provides an axiomatic treatment of it in line with Peano's. Dedekind writes this axiom as follows:

> The preceding theorem, as will be shown later, forms the scientific basis for the form of demonstration known by the name of complete induction (the inference from n to n+1); it can also be stated in the following manner: In order to show that all elements of the chain

[232] Ibid, 43.

> A_0 possess a certain property E (or that a theorem S dealing with an undetermined thing n actually holds good for all elements *n* of the chain A_0) it is sufficient to show
> ρ. that all elements *a* of the system A possess the property E (or that S holds for all a's) and
> σ. that to the transform n' of every such element n of A_0 possessing the property E , belongs the same property E (or that the theorem S, as soon as it holds for an element n of A_0, certainly must also hold for its transform n').[233]

The property E is held by every *n* and any element derived by application of the successor transform to *n*, in the case of the natural numbers *n* + 1. This is similar to the transform x+1 in Peano's axiom. Successor in Peano's terms must be put into transform in Dedekind's terms. And this does make sense, if a property is true for n and n+1, then that property must be held by the successor transform of n, n'.

Dedekind writes that if *m* and *n* are taken as numbers than one of the following situations is true:

1. $m = n$ $\quad n = m$ $\quad$ i.e. $m_0 = n_0$
2. $m < n$ $\quad n > m$ $\quad$ i.e. $n_0 \ni m'_0$[234]
3. $m > n$ $\quad n < m$ $\quad$ i.e. $m_0 \ni n'_0$[235]

We saw that in "Continuity and Irrational Numbers" Dedekind provided a construction of the real numbers. One instance in which this construction proves useful is in defining the signs <, >, and =. Dedekind defines these relations using cuts. In the real numbers a number *a* is greater than *b* if, for their cuts, there are two numbers such that they are members of A_1 and members of B_2. If there is only one number in A_1 that is in B_2 then the cut correspond to the same real number since with a cut of the rational numbers we are allowed to choose to which set the cut number belongs. This difference between A and B just corresponds to a difference in this choice and nothing more. Furthermore, if there are two or more members of B_1 that are members of A_2 then we say b

[233] Ibid, 61.
[234] ' here means successor, $n_0 \ni m'_0$ means *n* is a successor of *m*.
[235] Dedekind, *Essays*, 73.

is greater than a. Thus we see the power of the Dedekind cut. Mathematicians of this time were concerned with providing a rigorous foundation for the real numbers because the real number system played a larger role in mathematics than in any previous century.

Frege conceived of <, >, and = in the same way as Dedekind did. He defines equivalence through MUC. Frege conceived of < and > in terms of successors. Thus, A < B would be:

A ϕ C ϕ D ϕ E ϕ B

This would be for the natural numbers where ϕ = n+1. Dedekind writes almost the exact same thing, B is a member of the set of successors of A. This allows < and > to work for real numbers. Accompanying the arithmetic symbols in Dedekind's quote are included the corresponding set theoretical symbols. Frege wrote a similar passage in the *Begriffschrift*. Frege interprets A < B as B following A in the ϕ-series. Dedekind mentions in his letter to Hans Keferstein the pleasure of these similar treatments of the non-immediate successor. Peano states the same idea in *Principles*. In theorem 23 he writes: "a, b ε N: Ɔ : a < b. ∩ .a = b. ∩ .a > b."[236] In translation this states that if *a* and *b* are natural numbers then a is less than, equal to, or greater than b. These three thinkers, Dedekind, Frege, and Peano represent the mixing of logic and sets in the formation of arithmetic and numbers in handling <, >, and =. Frege, however, was not altogether sympathetic with Dedekind's program. Ferreiros tells that Frege criticized Dedekind on two counts.[237] The first was that Dedekind's notation was unwieldy. The second was that Frege accused Dedekind of neglecting the empty set. Ferreiros argues that the second criticism is unfair since Dedekind had introduced the empty set in an unpublished capacity, and ,in the conceptual way Frege would have approved of.

[236] Peano, "Principles", 118.
[237] Ferreiros, "Traditional Logic", 47.

Let us look at one more area of overlap that is a bit more interesting. This concerns just Peano and Dedekind. In Peano's chapter we saw his treatment of the Maxima and Minima. Peano puts forth the following three definitions:

> 1. If *a* is a class of natural numbers, then the greatest natural number of *a* equals an *x*, such that *x* is an *a*, and no *a* is greater than *x*.
> 2. If *a* is a class of natural numbers, then the least natural number of *a* equals an *x*, such that *x* is an *a*, and no *a* is less than *x*.
> 3. If *n* is a natural number and *a* is a class of natural numbers and *a* is not empty and there is no *a* such that it is greater than *n*, then the Maxima of *a* is a natural number.[238]

Dedekind uses the ideas of the Maxima and Minima for different purposes. He uses them as tests for infinitude. In articles 121, 122, and 123 of "The Nature and Meaning of Numbers"; he writes:

> 121. Every part *E* of the number-series *N*, which possesses a greatest number (111), is finite.
> 122. Every part *U* of the number-series *N*, which possesses no greatest number, is simply infinite
> 123. In consequence of (121), (122) any part *T* of the number-series *N* is finite or simply infinite, according as a greatest number exists or does not exist in *T*.[239]

In all three of these theorems the concept of the "greatest number" is used to test infinitude. This may remind the reader of Bolzano's discussion and what it had in common with Riemann's work. Bolzano and Riemann tackled infinity with the concepts of limitlessness and endlessness. In Dedekind and Peano we see a more rigorous approach.

As already mentioned many ideas were co-discovered by Dedekind and Peano. This is one place where logic crossed over into set theory. Peano was heavily influenced by Dedekind and mentions his debt to him in *Principles*." Peano writes in *Principles* that "The Nature and Meaning of Numbers" was great use to him. Syazhkin writes of Dedekind's influence on Peano that:

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[238] This is the author's translation of Peano's notation.
[239] Dedekind, *Essays*, 81, 83.

> It should be noted that Peano's axiomatization was to a significant degree inspired by the ideas expressed by R. Dedekind in his treatise. In general, Peano was under the strong influence of the theory of functions developed by Dedekind.[240]

Peano and Dedekind also had logicism in common, although Dedekind's philosophical proclivity has been a murky matter. This may be because there was not much philosophical activity in the set theory strain yet, and this may have changed because of Dedekind. Jose Ferreiros argues that Dedekind was a logicist. He points to the quote already mentioned in the introduction to this chapter. Ferreiros writes: "Among the earliest logicists we find Gottlob Frege (1848-1925), who restricted his logicism to arithmetic, and Dedekind, for whom all of pure mathematics was just logic."[241] Dedekind's logicism especially comes across in "The Nature and Meaning of Numbers." In this work Dedekind constructs set theory and then arithmetic on a logical foundation. This is the logicist's program, and we have seen it already in Frege and Peano. Dedekind's move into set theory was caused by his understanding of logic and the concept/set relationship. Ferreiros writes:

> This step toward set language, which Dedekind regarded as "natural," was difficult and strange for his contemporaries. What made it natural for Dedekind were, undoubtedly, two factors: his familiarity with the traditional logical conceptions, that established the concept/set relation; but above all his confidence in the approach through set theory to arithmetic and mathematics generally.[242]

Dedekind studied sets and their order because he felt that there was something fundamental in sets. He wanted set theory to be a foundational logic for mathematics. Ferreiros writes:

> Dedekind defines it (infinity) positively; the finite, instead, emerges as that which is not infinite. The reason for this lay in direct connection with Dedekind's aim of establishing set theory as the basis for mathematics, and in particular as the basis for a definition of the natural numbers.[243]

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[240] Styazhkin, *History of Mathematical Logic*, 279.
[241] Ferreiros, "Traditional Logic", 17.
[242] Ibid, 42.
[243] Ibid, 54.

This "set language" defined in a positive way what in mathematics had been taken negatively. He did the same with irrationals through cuts. This was a major part of Dedekind's logicist program.

Dedekind also engaged with mathematicians outside of set theory and logic. The mathematician Bernard Riemann was a contemporary and friend of Dedekind. Dedekind used Riemann's continuous manifold in his definition of continuity, i.e., real numbers, dense sets, etc. Ferreiros writes: "Riemann's influence on Dedekind is highly probable, since they were intimate friends from around 1855."[244] Riemann was a student at the University of Berlin early in the ascendency of Berlin as the new center of mathematics in Europe. Dedekind's influences highlight a period when Berlin was beginning to dominate European mathematics. It was the centrality of Berlin and Germany that led to the cross pollination between disparate fields. Another, even earlier example of this was Gustav Dirichlet. Dedekind was greatly influenced by the mathematician Dirichlet. Reimann, Dedekind, and Dirichlet composed a group of research mathematicians that existed at the University of Berlin and Gottingen early in those universities's reign over the mathematical world. Dedekind studied at Berlin shortly and received most of his education at Göttingen. Dedekind was the youngest of the three men. He studied under Dirichlet at Göttingen and even coauthored a textbook with him. On a sojourn to Berlin Dedekind met Reimann. After Dirichlet died Riemann took his chair at Göttingen. Dedekind was still studying at Göttingen at this time. He took classes with Riemann. After Dedekind graduated he stayed in Göttingen to lecture for a few years and was a colleague of Riemann's in the department. Dedekind cites Dirichlet multiple times in his articles. Riemann was attracted to the Berlin Faculty which included Dirichlet and likely took classes with him. Dedekind coauthored a textbook on number theory with Dirichlet, *Vorlesungen uber Zahlentheorie*. This work is cited throughout "Continuity and Irrational Numbers" and "The Nature and Meaning of Numbers."

[244] Ibid, 7.

One place where we see the effect of this work is in the definition of a number system like that of the real numbers. Dedekind writes:

> (the) completeness and self-contentedness which I have designated in another place as characteristic of a body of numbers and which consists in this that the four fundamental operations are always performable with any two individuals in *R*, i.e., the result is always an individual of *R*, the single case of division by the number zero being excepted.[245]

This passage from “Continuity and the Irrational Numbers” is taken almost verbatim from *Vorlesungen uber Zahlentheorie*.[246] This from the second edition of *Vorlesungen uber Zahlentheorie*, Dedekind states that this passage was written by him. We may safely guess that since the book was coauthored by Dirichlet that he was of a similar mind, though it should be pointed out that this was the second edition of 1871 whereas Dirichlet had died in 1859. Riemann was a lifelong friend and one-time teacher to Dedekind.[247] As mentioned in the Bolzano chapter Riemann advanced the concept of endlessness. Dedekind makes use of this same concept in articles 121, 122, and 123. These are quoted above. Dedekind was a close friend to Georg Cantor, covered in the next chapter. Thus a chain of influence can be formed from Dirichlet to Cantor.

Conclusion

Dedekind set up the achievements that will be outlined in the next chapter. He did this by creating a vocabulary for set theory and rigorously discussing concepts in membership and ordering. Dedekind’s achievements were heavily influenced by logic. Set theory was to be a foundation for mathematics. We saw that Dedekind’s work was shadowed by Frege and Peano. Dedekind read Frege and in a larger philosophic context Dedekind was won over to Frege’s

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[245] Dedekind, *Essays*, 5.
[246] Peter Gustav Lejeune Dirichlet and Richard Dedekind, *Vorlesungen über Zahlentheorie* (Braunschweig: F. Vieweg und Sohn, 1871), 424.
[247] This account can be found in Cajori’s History of Mathematics.

logicism. It was here that logicism found a home in set theory. The development of foundational mathematics provided an environment in which logic and set theory came together, and in the case of Peano and Dedekind, similar results were yielded. It will be seen in the next chapter that Georg Cantor worked in this environment and that set theory was born out of this environment.

7: Cantor

> I am so much for the actual infinite, that instead of admitting that Nature abhors it, as is vulgarly said, I defend that it affects her everywhere, in order to mark better the perfections of her Author. And so I believe there is no part of matter which is not, I do not say divisible, but actually divided; and consequently the least particle must be considered as a world full of an infinity of different creatures.
>
> -Gottfried Wilhelm Leibniz[248]

Introduction

In a sense, Georg Cantor (1845-1918) was a late-comer to the foundational environment outlined-in the previous chapter. But it was in his hands that set theory would reach its high mark in the nineteenth century. This chapter will argue for two theses:

> 1. Cantor was influenced by, and was the culmination of about seventy years of development to include Richard Dedekind, Gottlob Frege, Guiseppe Peano, and Bernard Bolzano.
> 2. Philosophical explanations for the impetus behind set theory are ultimately unsatisfactory. Cantor did not belong to any of the three schools of philosophy of mathematics that arose during his life. A more accurate statement is to say that Cantor was guided by theology not philosophy.

Cantor shared common concerns and ideas with the preceding mathematicians that have been discussed. Cantor used one-to-one correspondence as a criterion for saying that two sets have the same number of members. He also cofounded, with Dedekind, a vocabulary of set theory. Cantor's mathematics showed a great concern for the order of members in a set that I will argue came in part from Dedekind. Cantor, as well as Dedekind, created a vocabulary for concepts in set theory. The figures discussed in previous chapters that led to this high water mark all had

[248] Ferreiros, "Traditional Logic", 50.

attached to them philosophical biases which have been discussed. Cantor's philosophical views, however, are more complex than any other figure in this thesis. I do not believe Cantor was logistic. But, there can be no doubt that he was greatly influenced by logicism. The force of logicism is felt through his close friendship with Dedekind and his own reading of Frege. Instead of a philosophy of mathematics, I will argue that Cantor had a theology of mathematics. In making these points I hope also to capture what made Cantor's discoveries the climax of set theory. This is because in understanding these discoveries the reader can grasp the excitement of Cantor's ideas.

This chapter will begin with a brief discussion of Cantor's life.[249] Although in time he would be considered one of the greatest mathematicians Germany would ever produce, he was born in St. Petersburg. Georg Cantor was born in 1845 to Georg and Maria Cantor. When Cantor was a child the family moved to Heidelberg Germany. He attended the University of Berlin in the 1860's and finished his dissertation and habilitation in 1867 and 1869 respectively. Berlin by that time had become the center for mathematics research in Europe. While at Berlin, Cantor studied mostly under three professors: Karl Weierstrauss (1815-1897), Leopold Kronecker (1823-1891), and Ernst Kummer (1810-1893). After graduating with his doctorate he became a professor at the University of Halle. While his work was still quite distinct from mathematical logic, he was influenced by the logicism of thinkers like Frege. In Cantor we can see a mixture of logic and set theory. At the end of the nineteenth century investigations into systems of logic and infinity were increasingly coming into contact with each other. Cantor's work in set theory began in 1874 and continued in journal articles through the 1890's. The work I will be drawing on is *Contributions to the Founding of the Theory of Transfinite Numbers*, a work written in the 1890's. This work was originally published as two papers. These papers were the last significant work Cantor

[249] This account is taken from Dauben's *Georg Cantor* and Gratten-Guiness' *Search for Mathematical Roots*.

would do. Cantor died in 1918 amongst ever more frequent nervous breakdowns. This chapter will outline the accomplishments of Cantor in the field of foundational mathematics and connect him back to the previous mathematicians discussed.

The traditional approach to this history has neglected Cantor's predecessors. This can be seen in Jose Ferreiros' article, "Traditional Logic and the Early History of Sets, 1854-1908." In this work Ferreiros does discuss two previous mathematicians heavily; these are Bernhard Riemann and Richard Dedekind. He also mentions Frege and Bolzano, though he does not place the importance on them that I do. Ferreiros makes little mention of Peano and I have shown in at least one place where their investigations met one another. One of the aims of this chapter, as stated above, is to include Peano, Bolzano, and Frege in the story of the founding of set theory. In Cantor we see the finished product of the set theoretical strain. Cantor was engaging with both Bolzano and Dedekind. He used concepts that were central to Frege's understanding of sets. This finished product was naïve set theory. The paradoxes of naïve set theory would bring about an axiomatic treatment in the twentieth century. The genealogy of this can be considered to have taken off from Cantor. That is, the corpus of Cantor's set theory is a convenient break for our investigation.

Influences on Cantor

Cantor's accomplishments can be tied back to several of the mathematicians already discussed. Cantor was part of a group of students gathered around Karl Weierstrass (1815-1897) at the University of Berlin. They rediscovered the work of Bernard Bolzano. Weierstrass was Cantor's advisor. He held the position as the great old man of German mathematics. As mentioned above, because of Bolzano's troubles with the Czech authorities he was dismissed from academia. This led to Bolzano being unknown in mathematics for years. When it was finally rediscovered Bolzano's work had a great influence on Cantor. A large part of this influence probably came from Cantor's time with Weierstrass at the University of Berlin. Bolzano's influence can be seen especially in Cantor's founding proofs of set theory.

Cantor's work on set theory revolved around three proofs that he discovered in the 1870's. These dealt with the one-to-one correspondence between types of numbers. One-to-one correspondence was stated in Bernard Bolzano's Holomerism[250] and advanced further by Richard Dedekind. Cantor showed that the natural numbers can be put into a one-to-one correspondence with the rational numbers and also with the algebraic numbers. He also showed, surprisingly, that the natural numbers cannot be put into a one-to-one correspondence with the real numbers. This investigation into the natural numbers occurred chronologically parallel to Dedekind's, Frege's, and Peano's investigations into the natural and real numbers. Cantor's investigation, however, came from a different tradition than Frege's and Peano's. Cantor was in a line of mathematical research that included Bolzano and Dedekind.

Cantor's work was not isolated from the foundational environment discussed in the last chapter. He and Richard Dedekind had a long correspondence. Dedekind also carried on a correspondence with Bernhard Riemann (1826-1866) who was mentioned above. Dirichlet, who was also mentioned above as being important to Dedekind, was Riemann's dissertation advisor. Riemann, Dirichlet, and Dedekind were at Göttingen together. Thus, Cantor was the tail end of a chain of four mathematicians. This constitutes a second half of the great environment for mathematical research. Instead of reaching across both logical and set theory strains, these mathematicians were solely in the set theory tradition. This half of foundational mathematics was connected through Dedekind to the rest of the intellectual environment mentioned in the previous

[250] Holomerism was mentioned in the chapter on Bolzano so I will only mention that Bolzano himself defines it as a criteria for saying that a set has an infinite number of members.

chapter. It is worth pointing out that Peano briefly mentions Cantor in *Principles*, writing that his section on “Systems of Quantities” pertains “to the theory of those entities which Professor Cantor has called Punkmenge (sets of points).”[251] Ferrieros argues Cantor was influenced by Riemann. He considered sets from the point of view of sets of points. Riemann considered point-sets as an abstract way of investigating non-Euclidean spaces. He called these sets “manifolds.” Cantor would even occasionally use this term as a substitute for set.[252]Cantor’s set theory is the culmination of a thread starting from Dedekind’s concerns over infinity. This thread hits full force with Cantor. Dedekind’s influence can be seen in Cantor’s book *Contributions to the Founding of the Theory of Transfinite Numbers*. Cantor writes: “Thus ω is a number of the second number-class, and indeed the least. For if g is any ordinal number less than w, it must be the type of a segment of F_0.”[253] He also writes: “The second number class has a least number ω =lim v.”[254] Here we can see something like a Dedekind gap between the finite natural numbers which tend towards v and the least member of $\aleph_0$, ω. This is not to say that Cantor got this idea from Dedekind, but rather that although their ideas may seem disparate they are not. On a deeper level their ideas are related. Set theory arose from Cantor’s work in trigonometry, particularly in his consideration of point sets. These were sets of points which were important in Riemannian geometry and led Bernhard Riemann into a consideration of sets himself.

Cantor’s Discoveries

Cantor’s three proofs were spurred by the question: how do we determine that two groups of things have the same number of members?[255] This question was answered in an ultimately

[251] Peano, “Principles”, 102.
[252] Ferreiros, “Traditional Logic”, 26.
[253] Cantor, *Contributions*, 160.
[254] Ibid.
[255] This is not so much of a problem when discussing finite sets, the issue really only takes precedence when we talk about infinite sets and whether they have the same numbers of members.

unsatisfactory way by Bolzano, Cantor's handling of it was contemporary with Frege's and Dedekind's. The answer to this question was to use one-to-one correspondence to say that two sets have the same number of members. Cantor applied this criterion to infinite sets. He did this by finding ways to put the two sets into one-to-one correspondence, usually by a sort of instructions. If this was not possible Cantor would show how one-to-one correspondence between two sets could always be broken, again through a set of instructions. His founding of set theory centered on the discovery of three proofs in particular. The first proofs came in 1874[256]. These include the proof of one-to-one correspondence between the natural numbers and the algebraic numbers, and the proof that the real numbers cannot be put into a one-to-one correspondence with the natural numbers. This proof demonstrated that there were at least two degrees of infinity. If the real numbers were infinite and yet were a quantity not equal to the natural numbers, then there had to be at least two quantities of infinity. What Cantor showed is that if mathematicians took one-to-one correspondence as the definition for two sets having the same number of members then they would have to accept the unintuitive idea of two levels of infinity. This was true even for Bolzano's understanding of infinity. Cantor's third proof came in 1878,[257] and it showed that the rational numbers could be put into a one-to-one correspondence with the natural numbers. In all three of these proofs one-to-one correspondence plays a central role. This is exactly the distinction with previous conceptions of infinity. Earlier thinkers were of two opinions. Some believed that there was one kind of infinity, the infinity of natural numbers. Others, like Bolzano, accepted that there were different quantities of infinity, but they did not determine these by one-to-one correspondence. Bolzano held that, for example, the even natural numbers were of a lower infinity than the natural numbers. What Cantor realized was that the

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[256] The paper is: "Ueber eine Eigenschaft des Inbegriffes aller reelen algebraishen Zhalen." *Crelle's Journal* 77 (1874), 258-262.
[257] The paper is: "Ein Beitrag zur Mannigfaltigkeitslehre." *Crelle's Journal* 84 (1878), 242-258.

natural numbers and the even natural numbers could be put into a one-to-one correspondence, therefore they were equipotent. These proofs form the center of the corpus of naïve set theory. They achieved this by utilizing one-to-one correspondence.[258]

From the observation that there are two degrees of infinity Cantor created a way to count and calculate these infinities. Cantor founded the transfinite ordinal numbers. These are similar to the finite ordinal numbers: first, second, third, etc. They represent a place in the order of natural numbers. The sixth car is different from six cars. The former states that a car appears some place in an order, say the sixth car to pass a sign on the highway. The latter merely states that a total of six cars passed the sign. It says nothing about in what order they passed the sign. The transfinite ordinals begin with ω, the Greek letter omega. He states that ω is the limit that the natural numbers tend toward. That is, the natural numbers get closer, arbitrarily closer, to ω without ever reaching it. The historian of mathematics Phillip Johnson writes: "Cantor discussed the sense in which ω may be regarded as the limit to which the variable finite whole number n tends."[259] This idea of variable whole numbers is covered below; it is an important distinction for Cantor. This concept provides a better solution to the first paradox that we saw Bolzano treated. Ordinal numbers represent an abstraction from the objects they order. They give both order and number. A bucket of apples is abstracted to one degree when they are ordered, for instance the order in which they are picked out of the bucket. We go from saying apple, apple, apple, apple, apple, apple; to saying first, second, third, fourth, fifth, sixth. This constitutes an abstraction. Cantor even symbolized this as an M with one line over it. The line represents a single abstraction. Cantor writes of this:

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[258] A detailed treatment of these proofs would take too much room for this work. Also, these proofs are given in detail in many locations, for instance the following Phillip Johnson's *A History of Set Theory* p.23-24; and Raymond Wilder's *Introduction to the Foundations of Mathematics* 80-83, these works cover the algebraic number proof and the real number proof for the rational number proof se Howard Eves' *An Introduction to the History of Mathematics*.
[259] Phillip Johnson, *A History of Set Theory* (Boston: Prindle, Weber & Schmidt, 1972), 38.

> every simply ordered aggregate M has a definite ordinal type M'; this type is the general concept which results from M if we abstract from the nature of its elements while retaining their order of precedence, so that out of them proceed units which stand in a definite relation of precedence to one another.[260][261]

After the natural numbers are passed through the next number is ω the next after that is ω + 1. After each of the natural numbers is added to omega the next transfinite ordinal number is ω x 2. This operation can be carried out ad infinitum. ω is the "order type" of the natural numbers. An order type is the way a set is ordered, how one ascends through its members. Later in this chapter Cantor's concept of "similarity" will be discussed.[262] If two sets are similar then they have the same order type. The rational numbers and natural numbers are equal quantities, but they are not of the same order type. These ordinal numbers comprise a whole new arithmetic, separate from natural number arithmetic. It is worth going into this move that Cantor makes.

The transfinite ordinals and their arithmetic present one the most important moves that Cantor made in his system. To understand where they came from we should begin by setting aside transfinite ordinals and instead consider the following riddle. We have seen in previous chapters that the natural numbers can be placed into a one-to-one correspondence with the even natural numbers. This is accomplished with the function 2n. That is, we can get any even number by applying the function 2n to a unique natural number. For instance, the even number 467378 can be reached by applying 2n to the natural number 233689. Since one-to-one correspondence is the criterion for two sets having the same number of members, we can say that N (natural numbers)=E (even natural numbers). We can do the same thing with the odd natural numbers. We achieve this with the function 2n-1 (assuming 0 is not a natural number). We can reach the odd

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[260] Cantor, *Contributions*, 151-152 .
[261] I am writing M' in place of Cantor's M with a line over it, which what Cantor writes in *Contributions*.
[262] I will go ahead and share Cantor's definition of similarity here: We call two ordered aggregates M and N "similar" (ahnlich) if they can be put into a biunivocal correspondence with one another in such a manner that, if m_1 and m_2 are any two elements of M and n_1 and n_2 the corresponding elements of N, then the relation of rank of m_1 to m_2 in M is the same as that of n_1 to n_2 in N.

natural number 849 by multiplying 425 by two and subtracting 1. As before, we are able to say N=O (odd natural numbers). Now, how large is the set resulting from the addition of E to O? They are disjoint. Intuition tells that the addition of E to O is just the natural numbers. That is, E + O = N. However, by arithmetic, since N=E and N=O, E + O should equal 2N. That is, E added to O should result in a set having twice as many members as the natural numbers. What we are seeing here is that arithmetic is breaking down and leading to absurd results such as E + O = 2N. Cantor had to deal with this. To do this he invented the transfinite ordinal numbers and their arithmetic. E = ω and O = ω, so E + O = $\omega + \omega$ which equals N which equals ω. So in our new arithmetic (TOA) $\omega + \omega = \omega$.[263]

But Cantor's move to TOA should not be viewed as something he willed into existence. Cantor writes of the ordinal numbers:

> I was logically forced, almost against my will, because in opposition to traditions which had become valued by me in the course of scientific researches extending over many years, to the thought of considering the infinitely great, not merely in the form of the unlimitedly increasing, and in the form, closely connected with this, of convergent infinite series, but also to fix it mathematically by numbers in the definite form of a "complete infinite."[264]

The reader may sense in this passage that Cantor did not set out to found a transfinite arithmetic.[265] It is important to recognize that Cantor did not construct his proofs as a way to advocate a philosophical position, and we can imagine his being surprised by the firestorm it

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[263] I will take this opportunity to talk a little more about TOA. In TOA addition is not commutative in transfinite ordinal arithmetic. In traditional arithmetic 2+3 equals 3+2, the order of the numbers in not important. In TOA $3 + \omega$ is not the same as $\omega + 3$. It is important to state that the transfinite ordinals describe order. Order is about predecessors, what is the predecessor of a number. For example, the order of 1, 2, 3 depends on 1 being the predecessor of 2, and 2 being the predecessor of 3. $\omega+3$ can be written as the sequence 123456789…123. How many numbers in this sequence do not have predecessors? We see that the first "1" does not and that the second "1" does not. Now consider $3+\omega$. We can write this as 12312345…. Here we see that only one number in this sequence does not have a predecessor, the first one. Therefore $\omega+3$ and $3+\omega$ makes be seen to be saying different things. Cantor is making a move here by creating the transfinite arithmetic with ordinal numbers. However, an explanation would be technical. See the appendix for this explanation.

[264] Cantor, *Contributions*, 53.

[265] The term "transfinite arithmetic" refers to the arithmetic amongst the different levels of infinity that Cantor discovered. A simple example of this is: $\omega + 1 = \omega$.

cause. This fact may have bearing on events in his personal life that will be covered later. The role of mathematical rebel was forced upon Cantor by the inability to correspond the natural numbers and the real numbers. If these two number systems were both infinite but could not be put into one-to-one correspondence then by definition there had to be multiple levels of infinity. Cantor went still farther in positing an arithmetic with the ordinal numbers. While Bolzano was correct that an infinite set is one with no last term, Cantor's accomplishment is to speak of the first number after all the natural numbers. There is some limit that the natural numbers tend toward and Cantor instructs us to call it ω. In a previous chapter we saw that Bolzano and Riemann had different views of infinity. The distinction was made between infinity as endless and infinity as limitless. As I have previously mentioned, Cantor states that the infinity of natural numbers does have a limit, omega. In this sense it is like the equator of the earth. We can walk forever over the face of the Earth, nonetheless we are confined. Cantor viewed this as an uninteresting kind of infinity. The term "ordinal" suggests that order is important somehow. Ordinal numbers do not just tell us the number of things considered but suggest an order also. Omega is the ordinal after all natural numbers. We can think of omega as the ω^{th} number. Cantor calls the set of transfinite ordinals the second number class. Cantor posits another kind of transfinite number, the transfinite cardinal number.

Cantor sets out to prove that the cardinal number of the transfinite ordinals does not equal $\aleph_0$. It was mentioned earlier that Cantor's three proofs established that there are at least two levels of infinity. Cantor listed these levels off with cardinal numbers. A cardinal number tells the number of members in a set. Whereas the ordinal numbers represented one level of abstraction the cardinal numbers represented two levels of abstraction. As with our earlier example with apples, not only is the nature of the apples removed but also their order is removed and there are just six apples. The cardinal numbers of finite sets are just the natural numbers 1 2 3 4 … The cardinal numbers for the first two levels of infinity are symbolized as $\aleph_0$ for the natural numbers and $\aleph_1$ for the continuum. The cardinal numbers are different from Bolzano's degrees of infinity. In Bolzano's system the set of natural numbers was larger than the set of even natural numbers. The proof that the set of transfinite ordinal numbers does not have cardinality $\aleph_0$ runs thus.[266] So it must be a member of a next higher class of numbers. This is very likely the sort of argument that Leopold Kronecker would attack in his debate with Cantor. It is a reductio argument[267] and this is the sort of reasoning that intuitionists did not accept. Now let us discuss this new cardinal number that constitutes the totality of the transfinite ordinal numbers.

With his three proofs and the above proof Cantor shows that the real numbers form a more numerous degree of infinity and the transfinite ordinals form a higher degree of infinity. The real number system had increased in importance in mathematics in the nineteenth century. Still, little was known about what the real numbers were. In a previous chapter we saw that Dedekind went a long way in remedying this deficit. If, as Cantor showed, there were more real numbers than natural numbers this fact would seem essential for understanding the real numbers. As mentioned above the natural numbers could not be placed into a one-to-one correspondence with the real numbers. This infinity higher than $\aleph_0$ was named the continuum. Cantor spent much of his career trying to prove the Continuum Hypothesis. This states that the next transfinite cardinal number after $\aleph_0$ is the number of the continuum. This is the infinity of the transfinite ordinal numbers mentioned earlier and the real numbers.[268] There, of course, was recognition of

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[266] This proof can be found in full in the appendix.
[267] This sort of argument is also called "indirect proof", the way it works is that the negation of what you are trying to prove is assumed. At the end of a line of reasoning from this assumption a contradiction is reached. It is then shown that the negation of what we were trying to prove most itself be false. Therefore what we were trying to prove has been proven.
[268] This is one of the early proofs Cantor wrote and it is contained in the appendix.

the continuum before Cantor; one recalls the continuum paradox discussed in the Bolzano chapter. In Cantor's system both sets would have the same cardinal number, $\aleph_0$. Phillip Jourdain writes that Cantor dismissed Bolzano's continuum as covering only part of what a continuum is. Jourdain writes:

> Bolzano's (1851) definition of a continuum is certainly not correct, which is also possessed by aggregates which arise from G_n when any isolated aggregate is removed from it, and also in those consisting of many separated continua.[269]

It will be remembered that Bolzano defined the continuum in terms of neighborhoods around points. Cantor's point is that this is insufficient as a definition of the continuum. By this definition, we could say that the real numbers minus the natural numbers is still a continuum. Since Bolzano's definition would still hold.[270] But this is not what the mathematician wants from the continuum. This would cause a Dedekind gap and that is just what we cannot have in the continuum. Cantor's work in real numbers was an engagement with Dedekind and Bolzano. Dedekind was in close correspondence with Cantor, and Cantor no doubt knew Dedekind's work intimately. Bolzano was familiar to Cantor because Cantor was a member of the group of students around Karl Weierstrass who largely rediscovered Bolzano's work. Jourdain here states just a stronger connection between Cantor, Dedekind, and Bolzano. Cantor's greatest contribution to the understanding of the real numbers was the continuum hypothesis.

Cantor did more than construct mathematical proofs like those above, or count off infinities. He, along with Dedekind, created a technical vocabulary for talking about sets. This is just what Bolzano did not have. Cantor uses the concept of similarity. Similarity is a characteristic that holds between two sets. Cantor writes:

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[269] Cantor, *Contributions*, 72-73.

[270] The natural numbers in this case would be the "isolated aggregate". "isolated here" means that in the real number system a neighborhood can be described around a natural number that does not contain another natural number. G_n is described by Jourdain on p.71 as a "plane arithmetical space", think of this as all the possible points of the Cartesian plane; although any dimension space can be used.

> We call two ordered aggregates M and N "similar" (ahnlich) if they can be put into a biunivocal correspondence with one another in such a manner that, if m_1 and m_2 are any two elements of M and n_1 and n_2 the corresponding elements of N, then the relation of rank of m_1 to m_2 in M is the same as that of n_1 to n_2 in N.[271]

Two sets are similar if they can be matched in a one-to-one correspondence and if the order is the same in each set. The set of natural numbers and the set of even natural numbers are similar. A one-to-one correspondence can be drawn between the two sets and if we take two pairs, say {1,2} and {7,14}, one is the predecessor of seven and two is the predecessor of fourteen. The set of natural numbers and the set of rational numbers are not similar, in the pairs {1,1} and {3, ½} one is a predecessor of three but one is a successor of ½. In fact we could not match the natural numbers with the rational numbers and keep the relation of rank. What is fundamental, which is showing through, is that there is no fraction with which we can generate all the rational numbers from the natural numbers. What is more fundamental than this even is that there is no least rational number. Suppose we tried to pair the natural numbers to the rational numbers and wrote: (1,1/5)(2,1/4)(3,1/3)(4,1/2)(5,1/1). So far the two sets seem to be similar. But we can always say: Pair the next natural number, 6, with a rational number less than 1/5. Once this is done the similarity is destroyed. For any pairing like this we can add 1 to the denominator and pair that rational number with the next natural number and thus destroy similarity.

Cantor dove deeper into the order of sets with concepts like similarity. He, like Dedekind, was concerned with ordering within a set. If Cantor had never developed his three proofs above it is likely he would have made a name for himself just for his work on ordering. There are two kinds of ordering for Cantor, simply ordered and well ordered. Cantor defined a simply ordered set as:

> We call an aggregate M "simply ordered" if a definite "order of precedence" (Rangordnung) rules over its elements m, so that, of every two elements m_1 and m_2,

[271] Cantor, *Contributions*, 112.

> one takes the "lower" and the other the "higher" rank, and so that, if of three elements m_1, m_2, and m_3, m_1, say, is of lower rank than m_2, and m_2 is of a lower rank than m_3, then m_1 is of lower rank than m_3.[272]

Here Cantor states that a set is simply ordered if for any two members that are chosen one will precede or succeed the other, and, if for any three members that are chosen, say *a*, *b*, and *c*; if *a* precedes *b* and *b* precedes *c* then *a* precedes *c*. The first requirement sounds a lot like Peano's twenty-third postulate of subtraction.[273] Peano is identifying the natural number system as simply ordered. Peano, a logician is using a different language and tradition to talk about ideas that are very close to Cantor's. If $m_1 = m_2$ then we are not dealing with two elements as stipulated in the above quote. The second requirement of simple order is close to Peano's fourth axiom, it is often referred to as the transitive property, it states: If *a*, *b*, and *c* are natural numbers then, if *a* equals *b* and *b* equals *c* then we can say *a* equals *c*. [274] Cantor is requiring the transitive property from simple order. There is another kind of order and Cantor calls this well ordering. Well ordering is defined by Cantor as follows:

> We call a simply ordered aggregate F "well-ordered" if its elements f ascend in a definite succession from a lowest f_1 in such a way that:
>
> 4. There is in F an element f_1 which is lowest in rank.
> 5. If F' is any part of F and if F has one or many elements of higher rank than all elements of F'; then there is an element *f'* of F which follows immediately after the totality F', so that no elements in rank between *f'* and F' occur in F.[275]

A well-ordered set is one in which there is a first member, and such that any subset contains a first member.

Cantor did not just posit the above concepts; he also made use of them. The following proof is central to Cantor's concept of a well ordered set. In *Contributions* Cantor writes:

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[272] Ibid, 110.

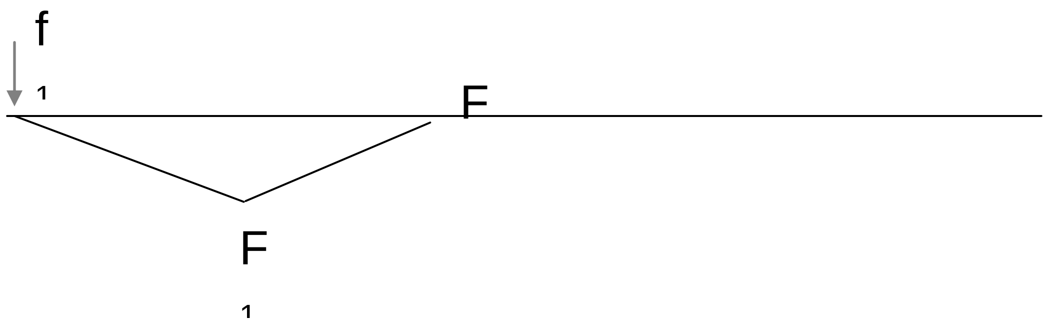
[273] Peano states this as a,b ε N : Ɔ : a < b.∪.a = b.∪.a > b in Peano, "Principles", 118.

[274] The transitive property is an argument of the same form as Barbara mentioned in a previous chapter. It can be written as a < b, b < c therefore a < c in arithmetic, or as all *a* are *b*, all *b* are *c*, therefore all *a* are *c*.. The transitive property is a fundamental kind of relation.

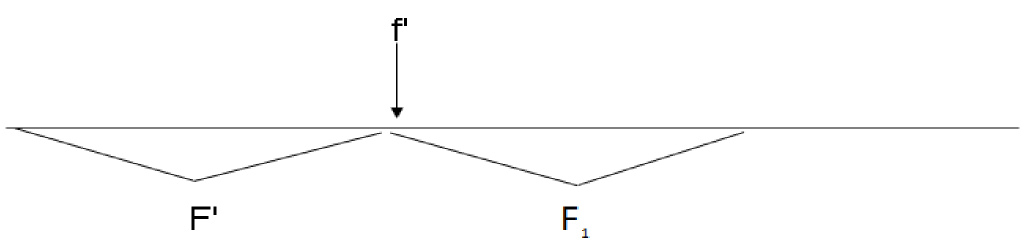
[275] Cantor, *Contributions*, 137-138.

> A. Every part F_1 of a well-ordered aggregate F has a lowest element.
> Proof.- If the lowest element f_1 of F belongs to F_1, then it is also the lowest element of F_1. In the other case, let F' be the totality of all elements of F which have a lower rank than all elements F_1, then, for this reason, no element of F lies between F' and F_1. Thus, if f' follows next after F', then it belongs necessarily to F and here takes the lowest rank.[276]

We can think of this proof in the following illustration:



This illustration shows the first state of affairs in which f_1 is the least member of F and a member of a subset F_1 of F. And so f_1 must be the least member of F_1. Now let us look at the second state of affairs:



Cantor directs us to consider the situation where f_1 is not a member of F_1, as in the above example. We imagine the set of all members of F that are less than all members of F_1. We denote this set above by F'. It is clear to see that there is no member between F' and F^1. In this case the next member in rank after F' must be the lowest member of F_1. This proof really comes out of

[276] Ibid, 139.

part two of Cantor's definition of well-ordered. This consideration is close to the Dedekind cut seen in the previous chapter. Dedekind's idea of a cut of the real number system relied on a similar partition of a set. Both Cantor and Dedekind were concerned with what happens when we investigate the members to either side of the partition. Is there such a member that we can identify if so or if not what does this say about the partition and the set?

As novel as Cantor's transfinite mathematics was, it dealt with fundamental questions and attempts that we saw in the chapter on Bolzano. Both Bolzano and Cantor asked how infinite sets were to be compared. As seen in a previous chapter Bolzano believed that the natural numbers were more numerous than the even natural numbers. Cantor on the other hand saw no difference between the size of the set of natural numbers and the size of the set of even natural numbers. Cantor's investigation of infinity took up where Bolzano's left off, except Cantor took one-to-one correspondence to its logical end. Cantor had to define the terms infinite and set. On the topic of infinity, Jourdain writes:

> The mathematical infinite, says Cantor, appears in two forms: Firstly, as an improper infinite, a magnitude which either increases above all limits or decreases to an arbitrary smallness, but always remains finite; so that it may be called a variable finite. Secondly, as a definite, a proper infinite, represented by certain conceptions in geometry, and, in the theory of functions, by the point infinity of the complex plane.[277]

Cantor divides infinity into two kinds. The first kind is what Dedekind called simply infinite. Bolzano discussed paradoxes dealing with the infinite and the infinitesimal. Cantor disregarded the infinitesimal. This infinity more closely reflects what Bolzano called infinity. This kind of infinity includes the infinitude of natural numbers tending toward ω. It does not include ω. This infinity increases past all limits because any limit posed in the form of a natural number can be passed by simply adding one to it. However, by the term improper it can be guessed that Cantor considers this infinity to be uninteresting. Because we can always name a larger finite natural

[277] Ibid, 55.

number this infinity is really just a variable finitude. The least transfinite ordinal omega is the least of the second number-class. It is these numbers that Cantor wants to investigate. Cantor defines a set as:

> By a manifold or aggregate I understand generally any multiplicity which can be thought of as one, that is to say, any totality of definite elements which can be bound up into a whole by means of a law.[278]

Here we see a definition of set that uses Frege's idea of the concept/set relation. It is the law or concept for Cantor that bounds the aggregate.

Cantor's Philosophy

The second thesis of this paper is pertains to Cantor's philosophy of mathematics. As mentioned above and throughout this chapter, Cantor was greatly influenced by Dedekind and there is analogy between the works of the two mathematicians. In the preceding chapter it was shown that Dedekind was logistic. Cantor, however, is a more complex matter. His set theory became a stage for the budding schools in philosophy of mathematics. This came out of the philosophical significance of set theory and the now famous debate with Leopold Kronecker (1823-1891). Kronecker was a professor at the University of Berlin while Cantor was a student there. The two were not strangers. Kronecker did not accept the methods which Cantor employed because he felt that they were spurious. This was because he held to a restrictive view on what can be said in mathematics. Kronecker held an exalted position in mathematics of the day. Cantor believed that Kronecker was using his position to block publication of Cantor's work. It is even suggested that it was this debate that exacerbated Cantor's mental health issues throughout his life. Kronecker is today considered to be a proto-intuitionist. It was he who famously said: " The integers were made by God, but everything else is the work of man."[279]

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[278] Ibid, 54.
[279] Wilder, *Introduction*, 192.

Intuitionism can be taken as the view that legitimate mathematics is that which appears to the categories of human experience or can be directly derived from those categories. Intuitionism would later gain great prominence under its most famous proponent L.E.J Brouwer (1881-1966) Cantor used methods that Kronecker considered suspect. One such method is the use of an indirect proof which we saw above with Cantor's proof that the power of the transfinite ordinals is not equal to $\aleph_0$. This debate brings to light the question of what Cantor's philosophical position actually was. Kronecker's objections would make Cantor nervous of stating his position too strongly, or stating the more novel parts of his theory in print. This was because of Kronecker's powerful position in the mathematical community at the time. What was fundamental to Kronecker's objections is that the proto-intuitionist position accepted, in a Kantian sense, only mathematical entities that made themselves apparent to categories.[280] This position was attacked by Frege in *The Foundations of Arithmetic*. The transfinite cardinal numbers certainly did not meet these criteria. This would build up to the last point in the history of mathematics where philosophy would throw its weight around in the arena of mathematics. Ferreiros makes the point that:

> Perhaps it is no coincidence that set theory was born in the land of rationalism, and precisely during the 19th century, when a close relation still reigned between scientists and philosophers.[281]

This history is outside the scope of the work. Cantor had to operate in a philosophically charged environment and it is important to consider his own philosophical position. Here, it must be admitted, that there is not much evidence for a logistic interpretation of Cantor. In fact, Jourdain

[280] To explain this further, Kant believed that all sense perception and reasoning was formed by categories. Mathematics was formed by the categories of time and space. Time gives rise to natural number arithmetic, space gives rise to Euclidean-geometry.
[281] Ferreiros, "Traditional Logic", 10.

states that in 1882 Cantor supported the "formalist theory of number."[282] This does not mean that Cantor was unaffected by logicism. Logicism was a view that held influence over the mathematicians whose work Cantor built his system upon.

As mentioned above, it is difficult to tell what Cantor's philosophical views were. He was interested in philosophy. But he was not particularly interested in the philosophy of mathematics. As stated above Jourdain ascribed formalism to Cantor at least when it came to defining numbers. Jourdain also attributes psychologism to Cantor, writing that Cantor: "defined "cardinal number" and "ordinal type" as general concepts which arise by means of our mental activity, that is to say, as psychological entities."[283] It should be mentioned here that Jourdain's opinions on these matters are valuable because, aside from producing the first translations into English of some of Cantor's works, he had a long correspondence with Cantor. However, in this period it is difficult to tell the difference between what might have been proto-formalism and what might have been logicism. There are three reasons for this. First, formalism was the last of the school of philosophy of mathematics to coalesce out of the nineteenth century. Formalism was in a nascent state at this time. Second, formalism constructed axiom systems much as the logicism did. The difference was what ontological state these axioms held. Were they logical necessities or were they merely chosen depending on what the mathematician wanted to accomplish? Opinions on these matters rarely found their way into mathematics texts. Finally, it is difficult to tell a proto-formalism from logicism because in these years it was really intuitionism that was the reactive agent in the philosophy of mathematics. This was seen above in Frege's *The Foundations of Arithmetic* and in Cantor's own debate with Leopold Kronecker. Intuitionism was the school of thought that the rest of the philosophy of mathematics, proto-formalism or logicism, had to

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[282] Cantor, *Contributions*, 70.
[283] Ibid, 202.

contend with. Rarely did the latter two schools do battle. Cantor's philosophy can be better explained by not appealing to any of these schools.

The issue of Cantor's philosophy of mathematics may be better explained in another way. In a discussion of Cantor's philosophical stance in mathematics it is useful to consider Cantor's personality. Historians of mathematics traditionally have not spent much time considering the personality or personal lives of mathematicians. The historian of mathematics Joseph Dauben writes:

> Historians of mathematics are generally accustomed to discussing ideas rather than individuals. A mathematician's personal life and his mathematics are frequently regarded as wholly separate, the former providing human interest while the latter comprises the heart of the matter.[284]

The view of historians of mathematics that Dauben gives us is one that I must admit to being sympathetic to. Dauben devotes much space to discussing Cantor's recurring nervous breakdowns. I do not wish to bat about Cantor's mental health in this work. Nonetheless, there is one area of Cantor's life that can illuminate the complex problem of what his philosophy of mathematics consisted in, if anything. I chose Leibniz's quote to start this chapter. It was quoted by both Bolzano and Cantor. Thus, this quote ties Cantor back to Bolzano who was the first figure in the set theoretical strain that we discussed. Leibniz's quote also points to a theological impetus for both Bolzano and Cantor. Thus, Leibniz's quote acts as bookends to the consideration of infinity from Bolzano to Cantor. Leibniz, Cantor, and Bolzano share a mathematical reverence for the "Author" of nature. This reverence will play a large role in Cantor's work. Cantor was the eldest child of Georg and Maria Cantor. His childhood was one of strict Lutheran upbringing. This was a pervasive theme in his life that his father in particular had

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[284] Joseph W. Dauben. "Georg Cantor: The Personal Matrix of His Mathematics." *Isis*, Vol. 69, No. 4 (Dec., 1978): 534.

bequeathed to him. Cantor's philosophical view of mathematics is difficult to tease out because his motivation was not philosophical but rather theological. He was motivated by a deep religious devotion. Imagine the "Author" that Leibniz spoke of in the quote at the beginning of this chapter. That passage was quoted by both Cantor and Bolzano. Dauben writes: "There can be no mistake about Cantor's identification of his mathematics with some greater absolute unity in God."[285] This may seem a poor substitute for logicism or intuitionism. But what Frege got from logicism was an impetus for foundational mathematics. This is just what religion provided Cantor. I think there is little reason to treat Cantor's religious views as inferior to logicism. This is a point of departure from the preceding figures we have discussed. Cantor's work built upon investigations that were largely guided by logicism, but he cannot not be counted as logistic.

Conclusion

In the preceding pages of this chapter we have seen that Georg Cantor was the climax of developments throughout the nineteenth century. He is responsible for five accomplishments. First, He used the one-to-one correspondence of Dedekind, Bolzano, and De Morgan more extensively and to greater effect than ever it had been used before. Second, he added to the vocabulary of sets that Dedekind had founded. Cantor's contributions were the concepts of well ordering, simple ordering, transfinite ordinals and cardinals, and similarity. Third, he fought against philosophical opinions of the day to publish his findings. Cantor's resoluteness provided incubation for the ideas of set theory. He was a wet nurse to set theory. The late nineteenth century was dangerous territory for new mathematical ideas. Mathematics in this period was more political than it had ever been. Kronecker held a powerful position in this environment and he turned his power against set theory, a lesser man than Cantor would have folded. Fourth,

[285] Dauben. "Georg Cantor", 547.

Cantor developed ideas of ordering that were more sophisticated than those of Dedekind or Bolzano. This would be of importance going further as mathematicians of the early twentieth century probed the well ordering of sets. Finally, He used these new concepts to construct the first generation of proofs of set theory. Among these were the three proofs of 1874 and 1878, and the well ordering proof mentioned above.

Conclusion

The prevailing history tells of the confluence of set theory and logic occurring in the early twentieth century. This treatise, though, has shown that mathematicians and logicians both were influenced by work and philosophical attitudes in the other tradition throughout much of the nineteenth century. This oversight may be due to the lack of shared language between the philosopher and the mathematician. The reader should recall Styazhkin's comment on the reception of Frege's *Begriffschrift* or Norbert Weiner's letter to Bertrand Russell. However, the foundational work of Frege and Peano was mirrored by Dedekind and Cantor. As shown above both Dedekind and Cantor read Frege. Both Frege and Peano were as concerned with building number systems as Dedekind and Cantor were. All four were concerned with how number systems and sets were constructed. These concerns stemmed from a shared philosophical viewpoint. With Cantor the philosophical viewpoint becomes nuanced so that he at least cannot, I think, rightly be counted as an outright logicist. Nonetheless, logicism was all around him.

Prior to these four thinkers there was a similar, if inchoate, situation. George Boole broke with the logicians discussed in the first chapter. Looking forward, he had an influence upon the work of Peano. Bernard Bolzano's work was the beginning of a modern treatment of infinity. He was also a crossover to logic with his magnum opus the *Wissenschaftslehre*. One-to-one correspondence has played a role in this narrative; for example, Frege's MUC, Bolzano's holomerism, Cantor's proofs, and Dedekind's definition of infinity. One-to-one correspondence linked the philosophy minded creators of mathematical logic with the founders of set theory. One recalls De Morgan's early description of one-to-one correspondence. The investigations of logicians came closer to set theory as the nineteenth century wore on; this is especially true of Frege and Peano. Peano, late in his career, wrote on set theory.

The Stanford Encyclopedia of Philosophy entry mentioned above states another aspect of the early history of set theory and that is "its role as a fundamental language and repository of the basic principles of modern mathematics." (Stanford Encyclopedia of Philosophy, s.v. "The Early Development of Set Theory") It was logicism that was the philosophical under pinning of the search for such a "fundamental language." The historian of mathematics Howard Eves writes of logicism:

> We have seen how these foundations were established in the real number system and then how they were pushed back from the real number system to the natural number system, and thence into set theory. Since the theory of classes is an essential part of logic, the idea of reducing mathematics to logic certainly suggests itself.[286]

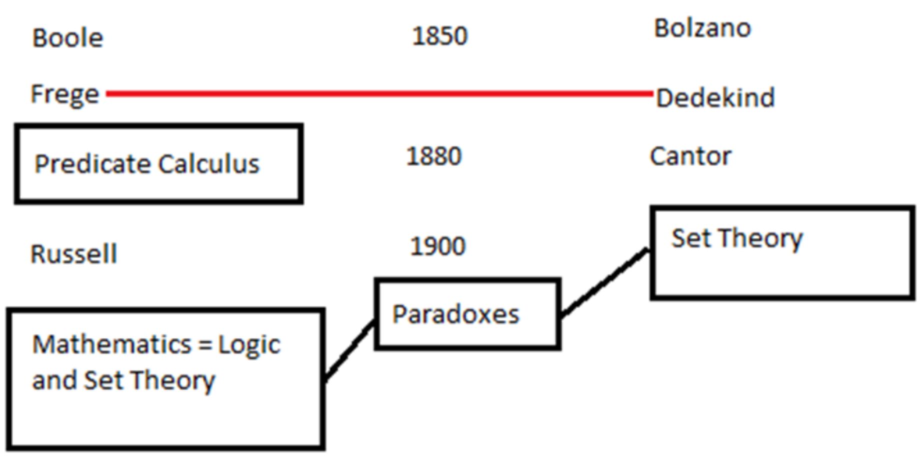
We have seen that the logic of Peano and Frege was meant to be a foundation for science. The condition of logic in the decades before the publication of *The Mathematical Analysis of Logic* was explained by the German philosopher G.W.F. Hegel when he wrote:

> In order that these dead bones of logic may be revivified by mind, and endowed with content and coherence, its method must be that by means of which alone logic is capable of becoming a pure science. In the present condition of logic, hardly a suspicion of scientific Method is to be recognized.[287]

Logic as a pure science is what Boole, Frege, and Peano offered. It may be difficult for the modern reader, living in a post-Gödel world, to see why these men were so excited at this prospect. What excited them was the prospect of deriving new scientific knowledge simply from a new combination of signs.

[286] Eves, *Introduction*, 629.
[287] GWF Hegel, *The Philosophy of Hegel* (New York: Modern Library, 1954), 189-190.

J.N. Crossely includes the following diagram in his book *What is Mathematical Logic?* Although Crossely is not an academic historian and his book is not scholarly, his illustration represents what has been the thinking on this history:[288]



I have inserted a line to indicate the point of greatest pre-1900 cross-pollination.

The reputation of these thinkers has increased since their lives. At the end of the nineteenth century W.W. Rouse Ball wrote *A Short Account of the History of Mathematics* (1888). Ball's book was a survey of the history of mathematics. What strikes the modern day reader is that although it was written after the work in logic and set theory covered here, there is little mention of any of the mathematicians here concerned. Raymond points this out:

> Take for instance the histories of Ball and Cajori, which were written shortly before 1900. In Ball's first edition (1888) there is no mention of "logic"; in the fourth edition (1908) there is a remark concerning George Boole to the effect that he "was one of the creators of symbolic or mathematical logic." The index of this book contains no citation to any other reference to logic. Cajori's first edition (1893) contains four remarks of a similarly incidental nature concerning logic.[289]

[288] Crossley, *Mathematical*, 2.
[289] Wilder, *Introduction*, 273.

Wilder then writes: “Compare these books with Bell’s *Development of Mathematics*, published in 1940. Here at least 25 pages are devoted to “mathematical logic.”[290] This is likely due to the split between mathematics and logic that was still existent in Ball’s and Cajori’s time. But more than this, it is likely due to the novelty and still hot mathematical debate over these ideas. Also, Ball’s work was *A Short Account of the History of Mathematics*, and the work of Peano, Frege, Dedekind, and Cantor were not history yet. Whatever the case may be, the subject is no longer ignored.

My criticism of the historiography has been that it has, to too great a degree, treated the logical as isolated from the set theoretical in the nineteenth century. The pathologies of this have been two; first, that histories of logic and set theory have ignored each other; second, that histories of logic and set theory have tended to work up to or up from Cantor and Russell. In this work I have made heavy use of Jean Van Heijenoort’s *From Frege to Gödel*. The most Van Heijenoort gives us of Cantor and Dedekind are letters. There are no papers from either author. It is not until paradoxes in set theory arose and the axiomatic treatment of sets that Heijenoort seems to offer his reader much material in set theory. But he does eventually offer it, which suggests that he feels that the investigation into the infinite be included in a volume on logic. *From Frege to Gödel* does include *Begriffschrift* and *Principles of Arithmetic*. But, as I have shown in the chapter on Dedekind, Frege and Peano make up only part of the special environment in the late nineteenth century. Heijenoort provides precious little material on Dedekind or Cantor. Rather, set theory is not seen as indispensable to a history of logic until the paradoxes of set theory arise. This thesis shows that logic has been a part of the history of set theory certainly by the time of Dedekind and perhaps further back to Bolzano. This thesis shows

[290] Ibid.

that set theory has been a part of the history of logic since De Morgan. Another work I want to mention, in order to exhibit the short comings of the historiography, is N.I. Styazkhin's book *The History of Mathematical Logic from Leibniz to Peano*. From the title it is clear that this work is building up to Bertrand Russell's *Principia Mathematica*. Styazkhin must have thought that Peano marked the end of an era. Styazkhin does not cover Cantor or Dedekind in this work. He ends his treatment of logic just before Crossley says that the two strains came together.

In the introduction I stated several areas were this history is important. I would like to add a discussion of one more here. In recent decades historians and philosophers of science have been occupied with a question. Thomas Kuhn, in *The Structure of Scientific Revolutions*, posited a large role for social factors in the development of science. Historians since have grappled with the extent to which science is a communal behavior just as religion or art are communal behaviors. This question has not penetrated so deeply into the history of mathematics. There are reasons for this. Mathematics is difficult to grasp without training and so there is little intuitive understanding in the public. Connected to this is the fact that there have been no great public debates in modern times involving mathematics. When the Christian world view in challenged in the west it is usually along two lines. These are first a debate about the age of the Earth, a debate with geology; and second, a debate about the origins of humans, a debate with biology. My thesis suggests an answer to the question of how social is mathematics. The answer to this question depends whether we are discussing mathematics internally or externally. Internally the answer is that the development of logic and set theory in the nineteenth century was certainly social in a Kuhnian way. First, an overwhelming majority of the figures discussed were university trained. Hence, we see the kind of initiation that Kuhn talked about. It was mentioned earlier that mathematics requires training and this fits well with Kuhn's idea in *Structure* that the science indoctrinates science students into the paradigm using training. We also saw that the debate between Kronecker and Cantor was really about what methods could be used in mathematics. Kronecker's proto-intuitionism could be seen to play the role of supplying mathematical morays, as a way of removing ideas that did not fit the community of research. Kronecker was able to do this because of his influence in this community. He was able to present so much opposition to Cantor's ideas because he was an editor at the most prestigious journal of mathematics at the time, *Crelle's Journal*. This has a very Kuhnian feel to it. Externally, this thesis answers a less satisfying "it depends." Certainly we saw a socially engaged and conscious logic in Britain. This logic had a social goal, that was, to improve the reasoning of the populace. German logic and set theory, however, were not nearly so engaged socially. Logic in Britain was a guide for the perplexed, logic (and set theory) in Germany was a foundation for the mathematician. The change in foundational mathematics in the nineteenth century was due to the change in what was desired.

For each of these thinkers I have spent much time on their systems and I should show what good this has done. The details of their systems were given above because it would be insufficient, in a work such as this, to gloss over the mathematics. This has been a fruitful approach. In the chapter on Dedekind a whole intellectual environment was largely constructed out of the details of the systems of Peano, Frege, Dedekind, and Cantor. In the chapter on Boole the reinterpretation of disjunction was a turning point in logic. This, largely, was a story of the details in De Morgan's, Boole's, Jevon's, and Peirce's systems.

The poet and literary critic Randall Jarrell once wrote that a great book "does a single thing better than any other book has ever done."[291] This may preclude me, by Jarrell's standards, from claiming greatness for the preceding work. If I have not done one thing better than anyone

[291] Randall Jarrell, *The Third Book of Criticism* (New York: FSG, 1965), 51.

ever has, I think I can claim to have done one thing that few have. That is, that I have paid due diligence to figures and years that have been neglected by historians.

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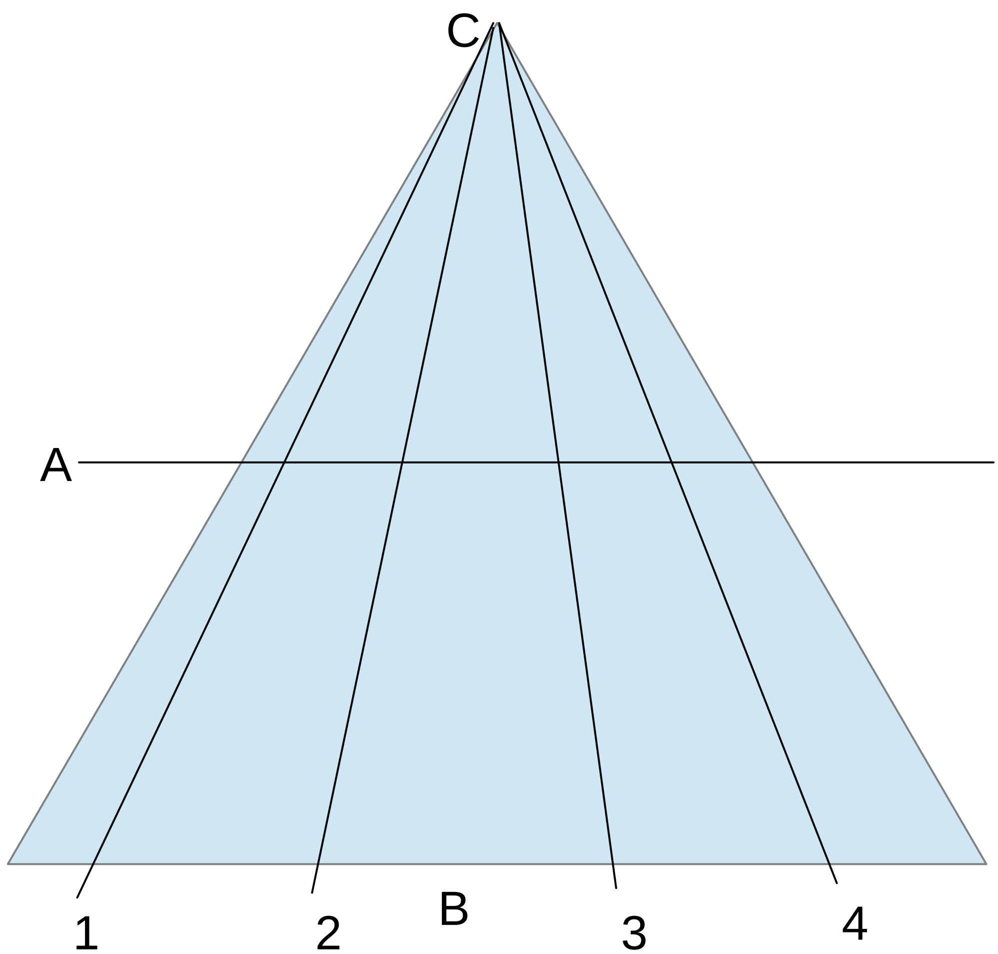
Appendix 1

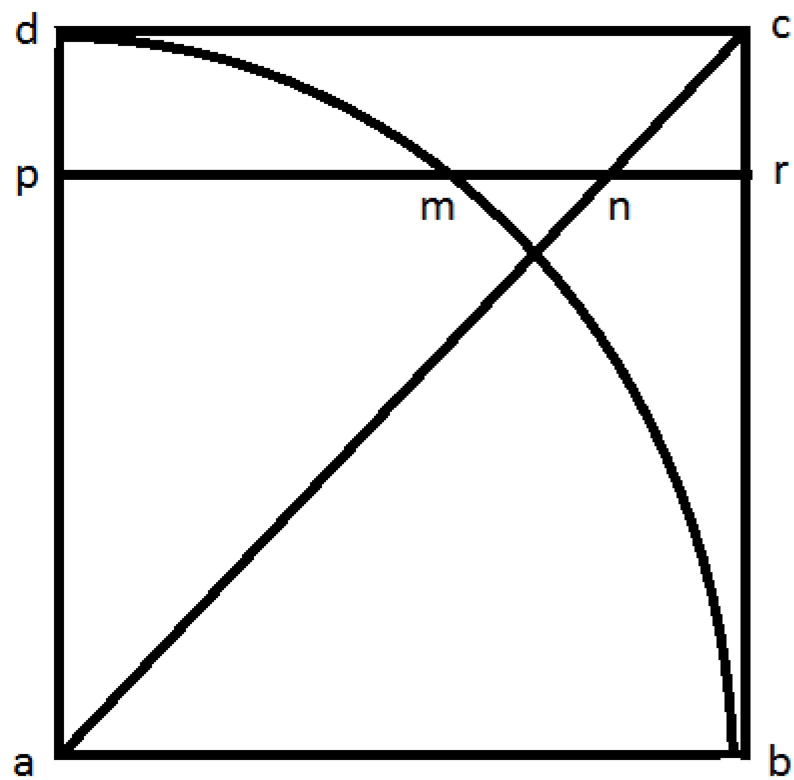
De Morgan’s one-to-one correspondence

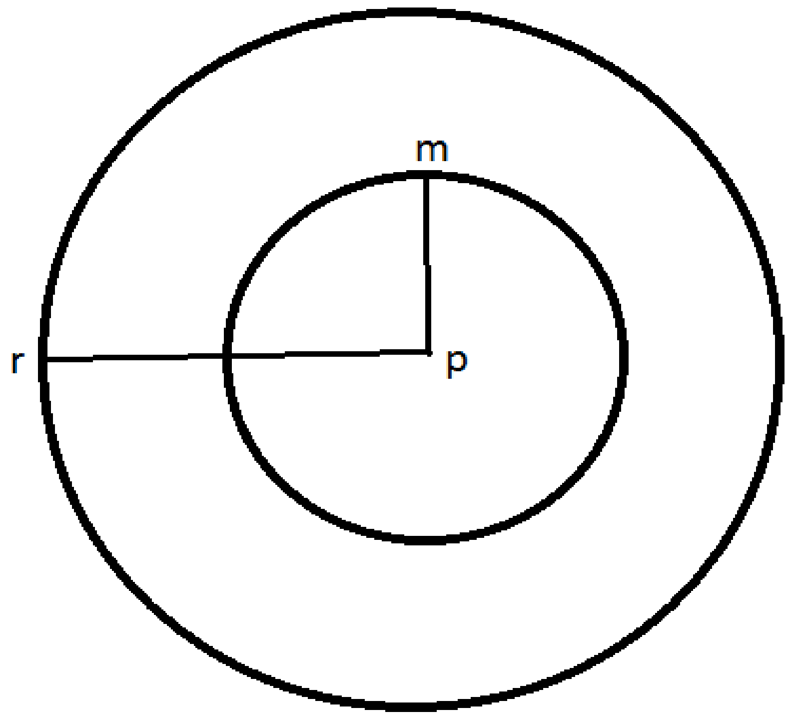


For every point on line B a corresponding point can be described upon line A simply by tracing a line from vertex C to line B. The intersection of this line at line A and then Line B will describe a unique pair of points that cannot be described by any other line.

Appendix 2

Bolzano's Paradoxes

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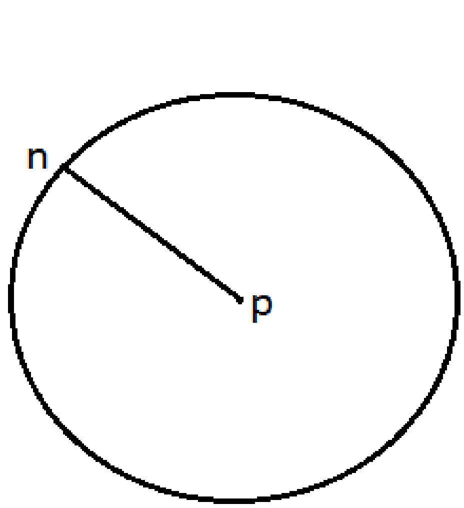
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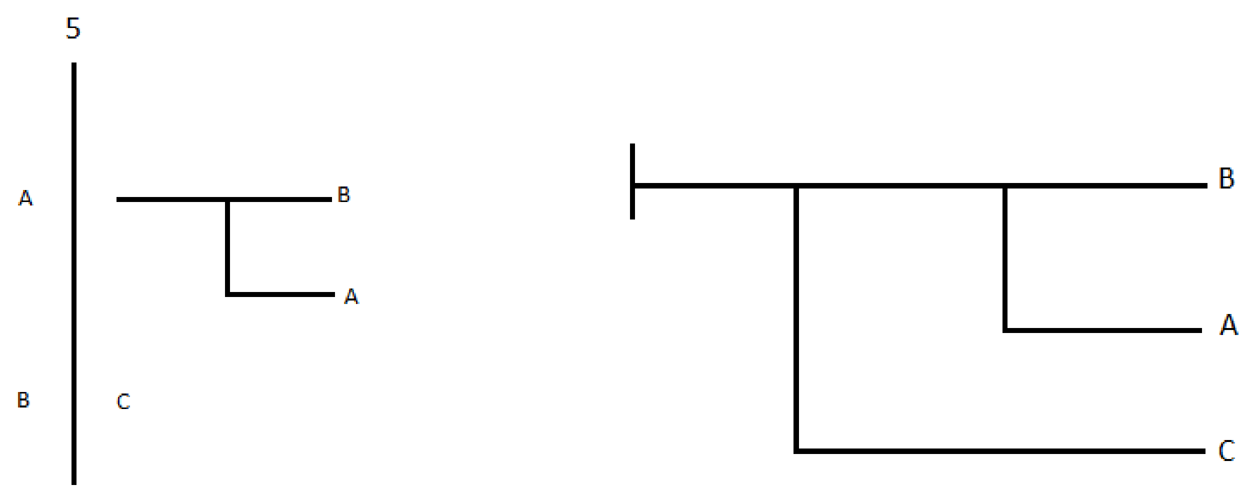
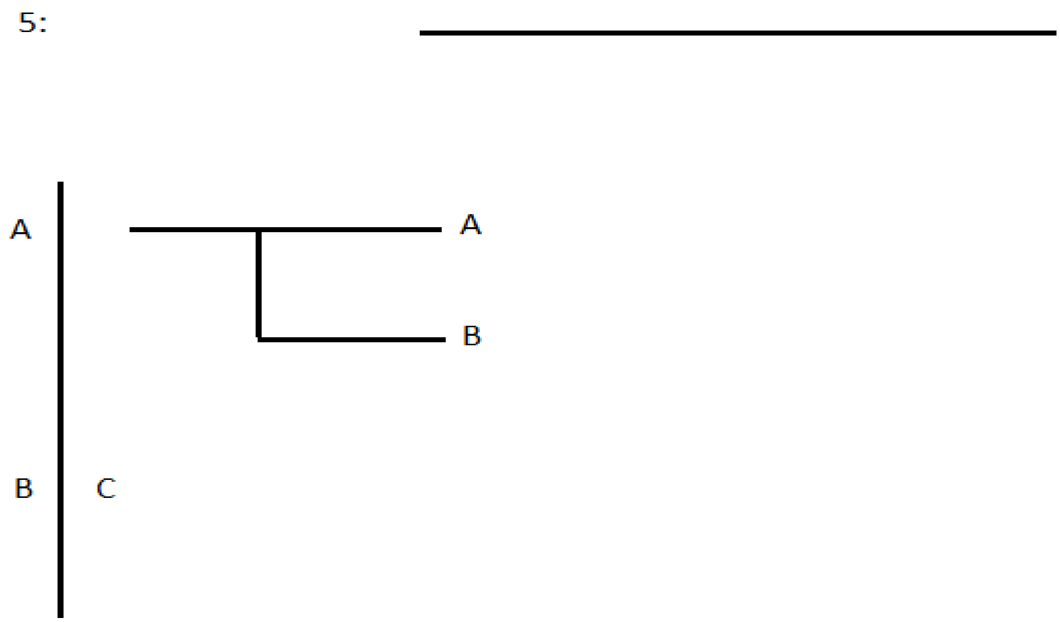
Appendix 3

Begriffschrift

In the *Begriffschrift* Frege gives the following notation to the left of his propositions:



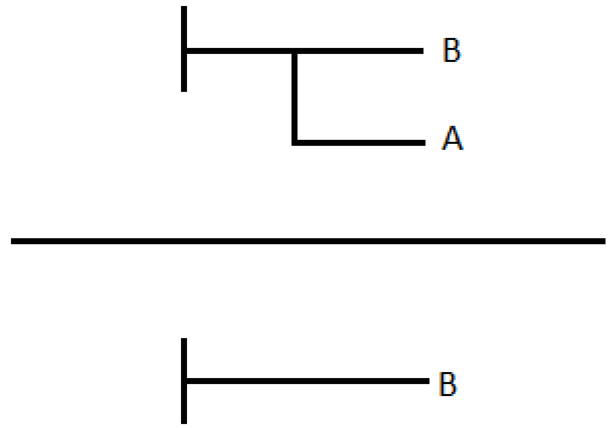
This notation means that the proposition to the left is derived by taking the proposition 5, referred to by the number above the vertical line; then replacing the letters on the left with the structures on the right. Another notation is shown below:



This notation means that the proposition above the line to the right is to be paired with the proposition 5, denoted by the number in parentheses followed by a colon. But first the proposition 5 is to receive the replacements noted below the number. In this case, in formula 5 *a* is to be replaced with “if *b* then *a*” and *b* is to be replaced with *c*. Once this is done the two formulas will yield, usually by modus ponens, the conclusion noted below the line.

Finally, Frege offers the following shorthand:

(XX):



Here the symbol XX is to be defined beforehand. In this case it is defined as the judgment *A*. It is then combined with the top judgment and with modus ponens produces the judgment *B*.

Appendix 4

MUC

In a previous chapter I mentioned the importance of MUC for Frege's understanding of the foundation of arithmetic. MUC is an integral part of this argument. Frege's full argument goes as follows. The concept, "identical to 0, and not identical to 0", has no number that falls under it. It is important to note that not even the number 0 falls under this concept. Because of this the number that belongs to the concept "identical to 0 and not identical to 0" is just the number 0, since no number falls under it. Now, imagine what happens when we remove the second half of this concept, we are left with just, "identical to 0." Here the number 0 falls under the concept, and so the number one belongs to it. Earlier Frege writes the definition of following immediately after, in the natural numbers. He writes:

> There exists a concept F, and an object falling under it x, such that the number that belongs to the concept "falling under F but not identical to x" is m is synonymous with n' follows in the natural numbers series immediately after m'.[292]

In the above example x is 0, F is "identical to 0", n is 1, and m is 0. With the distinction between "falling under" and "belonging to" Frege is able to get to the successor of 0. It is this way also that we see Frege can side step the criticism made that MUC itself contains the concept one, in the word "univocal." If Frege states the above argument first then he had constructed the number "1." He can then use it in MUC.

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[292] Frege, *Foundations*, 78.

Appendix 5

Peano's Correspondence

The difference between correspondence, similar correspondence, and reciprocal correspondence needs to be fleshed out here. Simple correspondence can be seen in this example:

A	B
1	1
2	2
3	6
4	2
5	4
	7
	4

Here every x of A has an f-correspondent in B. This is all that is meant by correspondence.

Similar correspondence we can see in the following example:

A	B
1	2
2	3
3	4
4	5

5	6
	7
	8
	9

Here we see that any two *x'*s in *A* will have the same rank as their *f*-correspondents in *B*. So, 2 and 4 in *A* have as *f*-correspondents 3 and 5 in *B*. Reciprocal correspondence can be seen in the following example:

A	B
1	2
2	3
3	4
4	5
5	6
6	7
7	8
8	9

Here we see that the correspondence is similar but also that every *x* in *B* now has an *f*-correspondent in *A*. Before, we were only concerned with the *f*-correspondents going from *A* to *B*. Now we go the other way. We see, of course that this means that *A* and *B* now have the same number of members.

Peano had a slightly different understanding of similarity. For Peano two sets were similar if any pair of unequal members of one set corresponded to a pair of unequal members of the other set. Of course this doesn't quite fit Cantor's spirit of similarity. By Peano's reckoning we can write that {A| $x<y$} is similar to {B| $fx>fy$}. Peano's similarity did not require that the two sets have the same number of members, that similarity goes both ways. He saves this for the reciprocal relation, which is closer to Cantorian similarity.

Appendix 6

Proof of the Irrationality of √2

First we must understand that a rational number is an integer over another integer, where the denominator cannot be zero. So let us write:

$$\sqrt{2} = a/b$$

We do not know what *a* or *b* is, we only suppose at the outset that *a*/*b* is in a fully reduced form. For example, 2/4 can be reduced to ½ but then we can no further reduce it. The first thing to notice about this equation is:

$$b\sqrt{2} = a$$

Next we square each side and we write:

$$2b^2 = a^2$$

Since *b*2 is by definition an even number we can say that a^2 is an even number. Since, if a square is an even number then its square root is also an even number. So then *a* is even and we may then write it as 2k. We now replace *a* in a^2 with $(2k)^2$. Carrying out this operation we get $4k^2$ and we write:

$$4k^2 = 2b^2$$

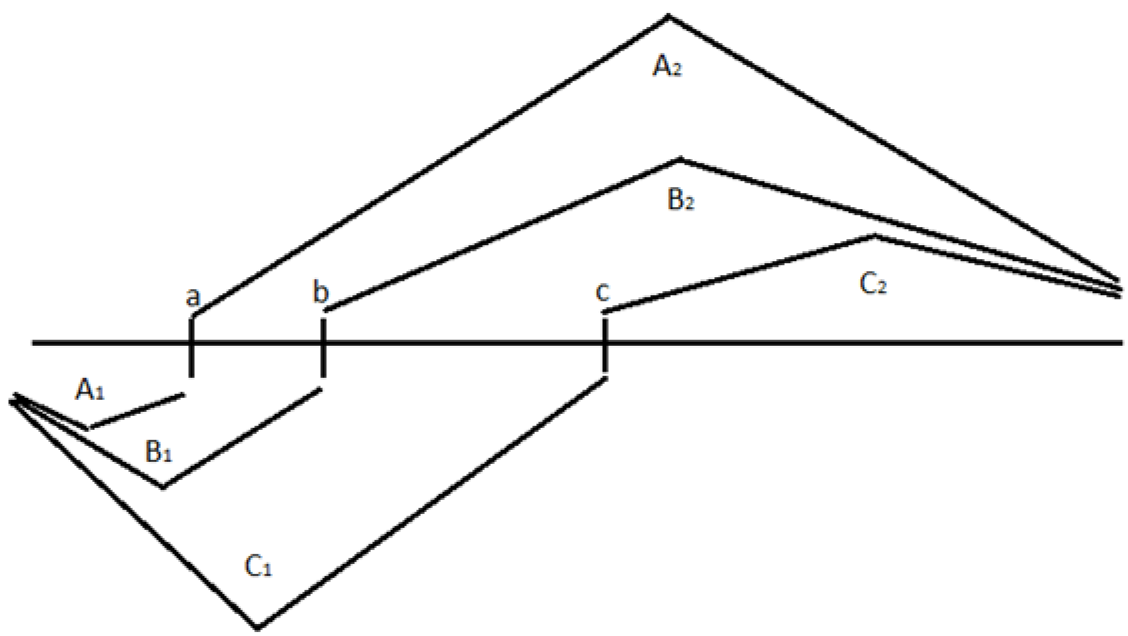
If we now divide each side by 2 we get:

$$2k^2 = b^2$$

Since $2k^2$ is of the form 2n we can say that it is even and so also that b^2 is even. And so b is even. What we did in this proof was assume that √2 was rational and we have arrived at a contradiction. We began the proof with the supposition that a/b was fully reduced. But we have also shown that *a* is even and b is even and so a/b cannot possibly be fully reduced since we could easily reduce it just by dividing the denominator and numerator by 2.

Appendix 7

Dedekind's Model for the Addition of Real Numbers



Appendix 8

Dedekind's Proof of the Existence of Irrational Numbers and their Definition

Dedekind tells us to imagine a positive integer *D* such that *D* is not the square of any integer. Imagine this number dividing all of the integers into two groups. The first group we will call A_1. This group will contain all of the integers; we will call them a_1, such that a_1^2 is less than *D*, or such that a_1 is a negative integer. In the second group we will call A_2. In this group we will put all integers; we will call them a_2, such that a_2^2 is greater than *D*.

It may not be clear from what we have said of *D* that $\sqrt{D}$ is not rational. After all Dedekind only stipulates that D is not the square of an integer, he says nothing of rational numbers. Dedekind gives us proof of this also, it is another reduction proof, it follows. Early in the proof Dedekind tells us that since *D* is not the square of any integer then we can state the following inequality: $\lambda^2 < D < (\lambda+1)^2$ where λ is some positive integer. What this is actually telling us is that the square root of *D* falls between some integer and its successor. So suppose that there is a rational number such that its square equals *D*. Dedekind instructs us that there are then two positive integers, *t* and *u*, such that: $t^2 - Du^2 = 0$. This has to be explained why exactly this is so. $\sqrt{D}$ is supposed to be a rational number. So if we take this rational number to be $t/u = \sqrt{D}$, then squaring each side of this equivalency gives us $D = t^2/u^2$. This relationship can be written as $Du^2 = t^2$. This can be written as $t^2 - Du^2 = 0$. Dedekind then tells us to assume that *u* is the smallest positive integer for which this is true. This is the assumption that will be contradicted eventually. You will recall at the beginning of the proof Dedekind writes $\lambda^2 < D < (\lambda+1)^2$. Dedekind now writes $\lambda u < t < (\lambda + 1)u$. This is gotten by multiplying λ^2 by u^2, D by u^2, and $(\lambda+1)^2$ by u^2. D multiplied by u^2 as we saw gives us t^2. We then have $\lambda^2 u^2 < t^2 < (\lambda + 1)^2 u^2$. If we take the square root of this inequality we get $\lambda u < t < (\lambda + 1)u$. We now create a number u' such that u' = t- λu. We create another number t' such that t' = Du- λt. By the inequality we see that u' is a positive integer since t, λ, and u are all positive integers and t is greater than λu. Also, t sits between λu and (λ+1)u. Therefore t sits somewhere in the space between λu and (λ+1)u this space is smaller than u. Therefore, (t – λu) < u, therefore u' < u . t' is also a positive integer. This can be seen by the fact that $t^2\text{-}Du^2=0$. This is just another way of writing D = t^2/u^2. This means we can write $\lambda^2 < t^2/u^2 < (\lambda+1)^2$. If we take the square roots we get $\lambda < t/u < (\lambda+1)$. Next we notice that $t/u - \lambda > 0$. Since t is a positive integer we can say $t(t/u - \lambda)>0$. This is equivalent to $t^2/u\text{-}\ t\,\lambda > 0$. Now, we recall that $Du^2 = t^2$. If this is so then we can divide each side of this equality by u and rewrite the denominator. We will have $(D\cdot u\cdot u)/u = t^2/u$. We next can cancel out the u on the left side and we are left with $Du = t^2/u$. So now, since $t^2/u\text{-}\ t\,\lambda > 0$, we can write Du - t λ > 0. Dedekind already stipulated that t' = Du - λt, so therefore t' is a positive integer. The final move in this part of the proof is to consider the following equivalency: $t'^2 - Du'^2 = (\lambda^2 - D)(t^2 - Du^2) = 0$. It may not be clear that: $t'^2 - Du'^2 = (\lambda^2 - D)(t^2 - Du^2)$. Consider this, by the definitions of t' and u' we can write $(Du\text{-}\ \lambda t)^2 - D(t\text{-}\ \lambda u)^2$ in place of the left side of the equivalence. Performing the exponents we get: $(D^2u^2\text{-}2Du\lambda t+ \lambda^2t^2)$. $D(t\text{-}\ \lambda u)^2$ can be rewritten as $(Dt^2 + 2Dt\lambda u - D\lambda^2u^2)$. Now we have $D^2u^2\text{-}2Du\lambda t+ \lambda^2t^2 - Dt^2 + 2Dt\lambda u - D\lambda^2u^2$. Because 2Dtλu appears twice, once as addition and once as subtraction, we can get rid of both and reduce this to: $D^2u^2 + \lambda^2t^2 - Dt^2 - D\lambda^2u^2$. Now, let us consider the right side of our original equivalence, $(\lambda^2 - D)(t^2 - Du^2)$. If we carry out the multiplication we get: $\lambda^2t^2 - \lambda^2Du^2 - Dt^2 + D^2u^2$. Since the exact same parts now occur on either side of the equivalency the equivalency is thus proven. Now comes the contradiction. If D is not the square of an integer but is the square of a rational number then there is a number of the form such that $D = t^2/u^2$; Where D, t, and u are both positive integers as we have set out originally. We can also write this equivalence as $t^2 – Du^2 = 0$. Now remember also that we said that *u* was the least number for which this equation would work. But look at the equivalency we have just proven: $t'^2 – Du'^2 = (\lambda^2 - D)(t^2 – Du^2) = 0$. Surely we can infer that $t'^2 – Du'^2 = 0$. And we determined that u' was less than u. This generates the contradiction that; u is both the least and not the least number for which $t^2 – Du^2 = 0$ is true. The supposition that D could be square of a rational number is proven impossible.

So then what kind of number is $\sqrt{D}$, it is irrational. This means that the square of every rational number is either less than or greater than D since as we have shown it cannot be equal to D. Dedekind ends his proof by showing that $\sqrt{D}$ forms a cut such that A_1 has no greatest number and A_2 has no least number. Dedekind gives us the equation $y = (x(x^2 + 3D)) / (3x^2 + D)$. With this equation we can take any member a_1 of A_1, we plug it in for x, and find another member a_1', that is y, that is between a_1 and D. Also, we can take any member that one wishes to say is the least member a_2 of A_2. We can plug this member in for x and get another member a_2', that is y, such that it is between a_2 and D. This is the definition Dedekind gives for an irrational number, one in which A_1 has no greatest number, and A_2 has no least number.

Appendix 9

Cantor's Proof that *c* does not have cardinality $\aleph_0$

The power of the totality {α} of all numbers α of the second number-class is not equal to $\aleph_0$.
Proof.-If {α''}[293] were equal to $\aleph_0$ we could bring the totality {α} into the form of a simply Infinite series

$$\gamma_1, \gamma_2, \ldots \gamma_v, \ldots$$

Such that $\{\gamma_v\}$ would represent the totality of numbers of the second number-class in an order which is different from the order of magnitude, and $\{\gamma_v\}$ would contain, like {α}, no greatest number.

$$\gamma_1, \gamma_{p2}, \ldots \gamma_{pv}, \ldots$$

such that

$$1 < p_2 < p_3 \ldots < p_v < p_{v+1} < \ldots,$$
$$\gamma_1 < \gamma_{p2} < \gamma_{p3} \ldots < \gamma_{pv} < \gamma_{pv+1} < \ldots,$$
$$\gamma_v \leq \gamma_{pv}.$$

By theorem C of S15, there would be a definite number δ of the second number-class, namely,

$$\delta = \operatorname{Lim}_{v} \gamma_{pv},$$

which is greater than all numbers γ_{pv}. Consequently we would have

$$\delta > \gamma_v$$

for every v. But $\{\gamma_v\}$ contains all numbers of the second number-class, and consequently also the number δ; thus we would have, for a definite v_0,

$$\delta = \gamma_{v0}$$

which equation is inconsistent with the relation $\delta > \gamma_v$. The supposition {α''} = $\aleph_0$ consequently leads to a contradiction.[294]

[293] As before '' is to represent what Cantor writes as two lines over α, this represents the power, or cardinal number, of α.
[294] Georg Cantor, "Selections from *Contributions to the Founding of the Theory of Transfinite Numbers" God Created the Integers* (Philadelphia: Running Press, 2005), 1020-1021.

Vita

Douglas Andrew Tuers is a native of Brevard North Carolina. He holds a Bachelors of Arts degree in History from the University of North Carolina at Greensboro. He then enlisted in the United States Army where he served for three and a half years. He then attended graduate school and received his Masters in Library Science from North Carolina Central University. He graduated from Appalachian State University with his Masters of Arts in History with a minor in Philosophy.